MATH-GA.2702.001 Fluid Dynamics

Lecture 9 - The Navier-Stokes equations

I) Newtonian fluids

1) Stress tensor

In a non ideal fluid, surface forces are not only normal to the surface, unlike an ideal fluid.
As we have seen in Lecture 2, surface forces can be conveniently represented with the stress tensor T_{ij}.
Since we know that pressure forces are surface forces, we may decompose

$$T_{ij} = -p\,\underset{\substack{\uparrow \\ \text{Kronecker delta tensor}}}{\delta_{ij}} + \tau_{ij}$$

where τ_{ij} is called the deviatoric stress tensor.

You will show in homework 7 that a stability argument involving the torque applied on an infinitesimal fluid element implies that T_{ij} is a symmetric tensor, and so is therefore τ_{ij}: $\tau_{ij} = \tau_{ji}$.

It is reasonable to assume that the surface stresses on the fluid element are associated with its deformation, and hence the velocity field $\vec{u}$. When one specifies the way in which τ_{ij} depends on $\vec{u}$, one determines what one calls its <u>rheology</u>.

There is a wide variety of fluids, with a wide variety of behavior, and hence a wide variety of rheologies. We will focus here on the simplest of the rheologies: <u>Newtonian viscous fluids</u>.

The bulk translation of a fluid with constant velocity produces no additional surface force. Therefore, the stress tensor τ_{ij} will be linked to <u>partial derivatives</u> of the velocity $\vec{u}$. Since the dominant contribution comes from the <u>first derivatives</u>, we focus on the simple case in which τ_{ij} is associated with the first derivatives $\frac{\partial u_i}{\partial x_j}$ <u>only</u>.

A Newtonian viscous fluid is one for which the stress tensor is <u>linear in the $\frac{\partial u_i}{\partial x_j}$</u>.

For Newtonian fluids, we may thus write

$$\tau_{ij} = A_{ijkl} \frac{\partial u_k}{\partial x_l}$$

where A_{ijkl} is a fourth-order tensor whose entries may depend on the local thermodynamic state of the fluid.

The last important characterization for τ_{ij} corresponds to the statement that the fluid is isotropic: there is no preferred direction in the fluid. The behavior of the fluid is independent of the choice of the coordinate axes.

This means that A_{ijkl} must be an isotropic tensor, i.e a tensor whose components take the same values whatever the choice of Cartesian coordinate systems.
We have already encountered isotropic tensors:
- Scalars are isotropic tensors
- The only second order isotropic tensors are of the form $k\delta_{ij}$, with $k \in \mathbb{R}$

It can be shown (we will not do it in this class) that the only 4th order tensors which are isotropic have the form:

$$A_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu \delta_{ik}\delta_{jl} + \gamma \delta_{il}\delta_{jk} \quad , \text{ with } (\lambda, \mu, \gamma) \in \mathbb{R}^3$$

We can compute

$$\tau_{ij} = \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu \frac{\partial u_i}{\partial x_j} + \gamma \frac{\partial u_j}{\partial x_i}$$

Now, remember that τ_{ij} must be symmetric. This is only possible if $\mu = \gamma$.

We can therefore write

$$\underline{\tau_{ij} = \lambda S_{kk} \delta_{ij} + 2\mu S_{ij}}$$

where S_{ij} is the strain rate tensor we introduced in Lecture 11, and we recognize $S_{kk} = \vec{\nabla} \cdot \vec{u}$, the volumetric strain rate.

The complete stress tensor Π_{ij} for an isotropic Newtonian fluid thus is:

$$\Pi_{ij} = -p\delta_{ij} + 2\mu S_{ij} + \lambda S_{kk} \delta_{ij}$$

with $p(\vec{r},t)$, $\mu(\vec{r},t)$, $\lambda(\vec{r},t)$ scalar quantities.

The expression above is often rewritten in a more convenient form by splitting the last two terms into:

$$2\mu S_{ij} + \lambda S_{kk} \delta_{ij} = 2\mu \left(S_{ij} - \frac{1}{3} S_{kk} \delta_{ij} \right) + \left(\lambda + \frac{2}{3}\mu\right) S_{kk} \delta_{ij}$$

where the tensor between parentheses on the right-hand side is traceless.

The quantity $\mu_v = \lambda + \frac{2}{3}\mu$ is called the coefficient of bulk viscosity.

A commonly made assumption is the Stokes assumption, corresponding to $\mu_v = 0$.

In that case,

$$\Pi_{ij} = -p\delta_{ij} + 2\mu\left(S_{ij} - \frac{1}{3} S_{kk} \delta_{ij}\right) \qquad \text{(Stokes assumption)}$$

μ_v tends to be important only when a fluid is being rapidly compressed or expanded, such as in sound and shock waves.

Another situation commonly encountered is that of incompressible flow, $\vec{\nabla} \cdot \vec{u} = 0$. In that case,

$$\Pi_{ij} = -p\delta_{ij} + 2\mu \, S_{ij}$$

Summary:

* $\Pi_{ij} = -p\delta_{ij} + 2\mu \left(S_{ij} - \frac{1}{3} S_{kk} \delta_{ij}\right) + \mu_v S_{kk} \delta_{ij}$ (Generic case)

* $\Pi_{ij} = -p\delta_{ij} + 2\mu \left(S_{ij} - \frac{1}{3} S_{kk} \delta_{ij}\right)$ (Stokes assumption)

* $\Pi_{ij} = -p\delta_{ij} + 2\mu \, S_{ij}$ (Incompressible flow)

for $p(\vec{r},t)$, $\mu(\vec{r},t)$, $\mu_v(\vec{r},t)$ scalars.

Fluid at rest

For a fluid at rest (possibly in a moving frame), $\tau_{ij} = 0$, so $\Pi_{ij} = -p\delta_{ij}$!

For a fluid at rest, there are no shearing stresses acting.

One-dimensional shear flow

Let us return to the shear flow in the x direction $u(y)$ seen in Lecture 1. We see that the only contribution to τ_{ij} is:

$$\tau_{12} = \tau_{21} = \mu \frac{du}{dy}$$

We thus recover Newton's law of viscosity seen in Lecture 1.

2) The Navier-Stokes equation

In Lecture 2, we derived the generic form of the momentum equation for a fluid with a general stress tensor Π:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = \rho\vec{g} + \vec{\nabla}\cdot\Pi$$

For a Newtonian fluid, the j^{th} component of the momentum equation therefore is:

$$\boxed{\rho\left(\frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i}\right) = \rho g_j - \frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\left(\mu_v - \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\right] + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\right]}$$

This is generally seen as the most general form of the Navier-Stokes equation, named after French engineer and physicist Claude Louis Navier (1785 - 1836) and physicist and mathematician George Gabriel Stokes (1819 - 1903) (the same Stokes as in Stokes' theorem).

Often, the temperature gradients in the fluid are small enough that it is a good approximation to treat μ and μ_v as constants. The Navier-Stokes equation then takes the more amenable form:

$$\rho\left(\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i}\right) = \rho g_j - \frac{\partial p}{\partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_i^2} + \left(\mu_v + \frac{1}{3}\mu\right)\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_i}\right)$$

(j component).

In vector form, this is:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = \rho\vec{g} - \vec{\nabla}p + \mu\vec{\nabla}^2\vec{u} + \left(\mu_v + \frac{1}{3}\mu\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})$$

For incompressible flow, this takes the particularly simple form:

$$\boxed{\rho\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} = \rho\vec{g} - \vec{\nabla}p + \mu\vec{\nabla}^2\vec{u}}$$

Incompressible flow and μ constant

This is a form of the Navier-Stokes equation very frequently used to study fluids in real applications!

3) Conservation of energy for a viscous fluid

In Lecture 8, we derived the following evolution equation for the internal energy of a viscous fluid:

$$\rho \frac{De}{Dt} = \vec{\nabla}\vec{u} : \Pi - \vec{\nabla}\cdot\vec{q}$$

With the general form for Π we obtained,

$$\rho \frac{De}{Dt} = -p\frac{\partial u_i}{\partial x_i} + \mu\left[\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \frac{2}{3}\left(\frac{\partial u_i}{\partial x_i}\right)^2\right] + \mu_v\left(\frac{\partial u_i}{\partial x_i}\right)^2 - \frac{\partial q_i}{\partial x_i}$$

This is of the form:

$$\rho \frac{De}{Dt} = -p\,\vec{\nabla}\cdot\vec{u} + \chi - \vec{\nabla}\cdot\vec{q}$$

where $\chi = \mu\left[\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \frac{2}{3}\left(\frac{\partial u_i}{\partial x_i}\right)^2\right] + \mu_v\left(\frac{\partial u_i}{\partial x_i}\right)^2$

can be interpreted as <u>the rate of heat generation per unit volume due to viscosity.</u>

Again, this equation agrees with the first principle of thermodynamics: the internal energy e changes due to the work $-p\,\vec{\nabla}\cdot\vec{u}$ and heat addition/subtraction due to viscosity and heat conduction.

II) Boundary conditions

Unlike the Euler equation, the Navier-Stokes equation involves second order partial derivatives of $\vec{u}$ with respect to the spatial variables, so we will need two boundary conditions on $\vec{u}$.

The choice of boundary conditions corresponding to a well-posed problem remains an open mathematical problem, but certain boundary conditions have been verified experimentally for a wide variety of fluids and regimes.

It is clear that the condition on the normal component of the velocity we had for the Euler equation still holds:

$$\vec{n} \cdot (\vec{u} - \vec{u_s}) = 0$$

The question is: what should be the boundary condition for the tangential components of $\vec{u}$?

* Common choice # 1: the no-slip boundary condition

The no-slip boundary condition states that the tangential components of fluid velocity at a rigid boundary must be equal to those of the boundary itself. This makes sense from a physical point of view: due to viscosity, the fluid is dragged by the boundary.

The full (normal + tangential) no-slip boundary condition thus is:

$$\vec{u} = \vec{u_s}$$

No slip boundary condition for viscous fluids

* Common choice # 2: Zero tangential stresses

At a free surface, such as a water-air interface, where the shear stresses exerted by the air on water are very small, the tangential boundary conditions are replaced by the condition that the tangential stresses are zero. This leads to conditions on the derivatives of $\vec{u}$.

For example, for the simple shear flow $\vec{u} = \langle u_1(y), 0 \rangle$, Newton's law of viscosity leads to the following zero stress condition at the interface:

$$\tau = \mu \frac{du_1}{dy} = 0 \quad \Rightarrow \quad \frac{du_1}{dy} = 0$$

III] Dimensionless form of the Navier-Stokes equation

1) Dimensionless form of the Navier-Stokes equation

As we did for ideal fluids in Lecture 3, we derive a dimensionless form of the Navier-Stokes equation by introducing the following dimensionless variables:

$$\bar{\vec{r}} = \frac{\vec{r}}{L}, \quad \bar{t} = \frac{t}{T}, \quad \bar{\vec{u}} = \frac{\vec{u}}{U}$$

We focus on the Navier-Stokes equation for μ constant and incompressible flow for illustrative purposes, since the equation can be written in an elegant, compact form. Likewise, for simplicity we consider a fluid with constant and uniform density.

Defining $\bar{p} = \frac{p}{\rho U^2}$, $\vec{g} = g\vec{e}_z$, the Navier-Stokes equation takes the form:

$$\left(\frac{L}{TU}\right)\frac{\partial \bar{\vec{u}}}{\partial \bar{t}} + \bar{\vec{u}} \cdot \bar{\vec{\nabla}}\bar{\vec{u}} = \frac{gL}{U^2}\vec{e}_z - \bar{\vec{\nabla}}\bar{p} + \frac{\nu}{UL}\bar{\vec{\nabla}}^2\bar{\vec{u}}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity.

- We recognize the Strouhal number $St = \frac{L}{TU}$ already discussed in lecture 3, and the Froude number $Fr = \frac{U^2}{gL}$ also discussed then.

- With viscosity, we have a new dimensionless quantity:

$$Re = \frac{UL}{\nu}$$

called the Reynolds number.

The Reynolds number measures the relative importance of inertia as compared to the viscous term:

$$\frac{\vec{u}\cdot\vec{\nabla}\vec{u}}{\nu\vec{\nabla}^2\vec{u}} \sim \frac{\frac{U^2}{L}}{\nu\frac{U}{L^2}} \sim \frac{UL}{\nu} = Re$$

Dropping the bars on the dimensionless quantities for the simplicity of the notation, the dimensionless form of the Navier-Stokes equation for a fluid with constant and uniform density and viscosity is:

$$\boxed{St\,\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} = -\vec{\nabla}p + \frac{1}{Fr}\vec{e_z} + \frac{1}{Re}\vec{\nabla}^2\vec{u}}$$

Of all the dimensionless parameters appearing in the equation above, the Reynolds number Re plays the most important role, as we discuss next.

2) Reynolds number and flow regime

- High Reynolds number flow: $Re \gg 1$

In the limit $Re \gg 1$, the fluid behaves mostly like an ideal fluid: viscosity does not have a strong effect on the behavior of the fluid.

This is true except in the vicinity of boundaries, where boundary layer behavior has to be considered. For these layers, the scale length L of the experimental set up is not the right scale length, which can be shown to be of the order:

$$\frac{\ell}{L} = O\left(\frac{1}{\sqrt{Re}}\right)$$

where ℓ is the scale length of the boundary layer.

As one would expect physically, the larger Re, the smaller the boundary layer.

This small boundary layer can however play a macroscopic role, as we saw when we discussed aerofoil theory: viscosity allows a boundary layer to detach from an interface, and vortex shedding, for example.

$Re \gg 1$ flow can thus be quite different from purely inviscid flow, even macroscopically.

Another key aspect of $Re \gg 1$ flow is that steady flow at high Reynolds number is often unstable to small disturbances, and the flow may become turbulent, as we will briefly discuss in the next lecture. This is one of the most challenging aspects of $Re \gg 1$ flow.

- Low Reynolds number flow: $Re \ll 1$

Low Reynolds number flow tends to be well ordered, laminar. Indeed, the viscosity is so large that it quickly damps out any turbulent behavior.

$Re \ll 1$ flow is characterized by its reversible nature, as we will see with an impressive video of an experiment with glycerine in class. This reversibility of the flow has implications for the swimming techniques of organisms swimming in high viscosity fluids, as we will briefly discuss in class.

$\boxed{\text{IV}}$ Vorticity equation

We conclude this lecture by looking at the modification to the vorticity equation for a viscous fluid.

We focus on a viscous fluid with constant and uniform density and viscosity, whose evolution is given by:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\frac{1}{\rho} \vec{\nabla} p + \vec{\nabla} \chi + \nu \vec{\nabla}^2 \vec{u}$$

where as in Lecture 4 we assumed that all the volume forces can be written as the gradient of a potential χ.

If $\nu = 0$, we have seen that taking the curl of the momentum equation leads to:

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \vec{\nabla}\vec{u}$$

Let us evaluate the curl of $\nu \vec{\nabla}^2 \vec{u}$ to see how viscosity modifies this equation.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \vec{\nabla}^2 \vec{u}$$

so for our incompressible flow,

$$\vec{\nabla}^2 \vec{u} = - \vec{\nabla} \times \vec{\omega}$$

Thus,

$$\vec{\nabla} \times (\vec{\nabla}^2 \vec{u}) = - \vec{\nabla} \times (\vec{\nabla} \times \vec{\omega}) = \vec{\nabla}^2 \vec{\omega}$$

We conclude that the vorticity equation for a viscous fluid with constant and uniform density and viscosity is:

$$\boxed{\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \vec{\nabla}\vec{u} + \nu \vec{\nabla}^2 \vec{\omega}}$$

$\nu \vec{\nabla}^2 \vec{\omega}$ is the rate of change of $\vec{\omega}$ caused by diffusion of vorticity. It is responsible for the breaking of Kelvin's circulation theorem for viscous fluids.