The purpose of this lecture is to motivate *variational formulations* of classical mechanics and to provide a brief reminder of the essential ideas and results of the calculus of variation that we will use in the next few lectures.

1 Classical Mechanics of Discrete Systems

1.1 Description of the physical system

1.1.1 Classical mechanics of discrete systems

The subject of the next few lectures will be the mathematical formulation and the study of the physical behavior of *discrete* systems as described by the branch of physics known as *classical* mechanics. *Discrete* means that the system is made of a finite number of elementary parts, and this number will often be quite small. Discrete is in opposition here to a branch of classical mechanics known as *continuum mechanics*, which is often applied to the study of solids and itself is a very rich and interesting field of applied mathematics. *Classical* means that we will not consider quantum effects.

1.1.2 Configuration space

The elementary parts forming the discrete systems are often called *point masses*. A point mass is a point particle, i.e. a single point in space, with a mass but no internal structure. We will then view extended bodies as composed of a large number of these elementary parts with specific spatial relationships among them. This is obviously an idealization, which will nevertheless be satisfying for all the situations we will cover in this course.

Consider for example the double pendulum system shown in Figure 1. If the masses of each bob are much larger than the masses of the rigid rods holding them, we can just view the system as made of two point masses, with masses m_1 and m_2, localized at the center of gravity of each bob, whose motion in space and with respect to one another is constrained by the fact that the bobs remain attached to the rods at all times.

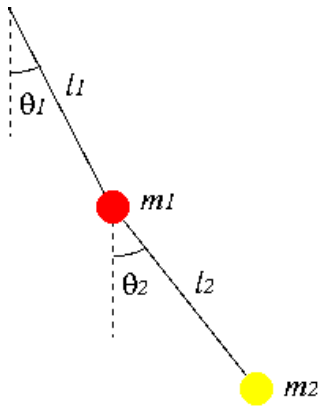

Figure 1: Double pendulum system

Specifying the position of all the constituent particles of a system specifies the configuration of the system. The existence of constraints among parts of the system means that the constituent particles cannot assume all possible positions. The set of all configurations of the system that can be assumed is called the *configuration space* of the system. The dimension of the configuration space is the smallest number of parameters that have to be given to completely specify a configuration. This number is commonly called the number of *degrees of freedom* of the system.

Consider the motion of a point mass in free space subject to some force $\mathbf{F}$. Three quantities are required to specify its position at any given moment in time. The number of degrees of freedom of that system is 3.

Consider the spring mass system shown in Figure 2. The system is usually considered in the idealized framework in which the motion of the ball with mass m is constrained to only be in the vertical direction. In other words, it is understood that the initial conditions and the rigidity of the spring are such that the motion is along the z axis. In that case, the system has one degree of freedom.

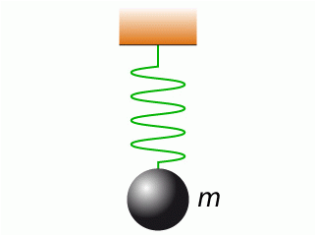

Figure 2: Spring mass system

The parameters that are used to describe the configuration of a system are called the *generalized coordinates*. For a complete description of a system, one needs at least as many generalized coordinates as there are degrees of freedom in the system. Depending on how one chooses to describe the system, one may however use *more* generalized coordinates than there are degrees of freedom.

If we go back to the double pendulum example shown in Figure 1, one may at first think that a convenient way to study the system is to take the Cartesian coordinates of each of the two masses as generalized coordinates, and to keep in mind, while setting up the equations describing the system, that there are constraints on the system that limit the possible configurations, since the masses m_1 and m_2 are not allowed to move independently from each other. With such an approach, the number of generalized coordinates would be 4, and there would be two constraints, namely the fact that the lengths l_1 and l_2 of the rigid rods are fixed.

After further thought, however, it appears that since the system has two degrees of freedom, it is more convenient to take the angles θ_1 and θ_2 as generalized coordinates. Sets of coordinates with the same dimension as the configuration space are generally easier to work with because we do not have to deal with explicit constraints among the coordinates. In this class, we will learn how to deal with both situations in an effective manner.

Now that we have seen how to describe a system at a given moment in time, we are ready to learn how to determine the motion of the system as a function of time. In classical mechanics, there are at least two equivalent ways of specifying the physically relevant path of the system in configuration space: either through *differential equations* known as Newton's laws, or through *integral equations*, associated with a variational formulation of classical mechanics often referred to as Lagrangian mechanics. Both approaches have separate merits, that we will discuss in detail. Since you are more likely to have been exposed to Newtonian mechanics at this stage of your academic career, we will start with a review of the Newtonian approach.

1.2 Newtonian approach to classical mechanics

1.2.1 Newton's law

Consider N point masses $m_{i,i=1\dots N}$ whose positions at any moment in time are given by the position vectors $\mathbf{r}_i(t) = (x_i(t), y_i(t), z_i(t))$ in Cartesian coordinates, and are subject to the forces $\mathbf{F}_i$ that may depend on time, the positions of the point masses $\mathbf{r}_i(t)$ and their velocities $\dot{\mathbf{r}}_i(t)$. Newton's law says that the motion of the point masses as a function of time is given by the vector equation

$$m_i \ddot{\mathbf{r}}_i(t) = \mathbf{F}_i(t, \mathbf{r}_1(t), \dots, \mathbf{r}_N(t), \dot{\mathbf{r}}_1(t), \dots, \dot{\mathbf{r}}_N(t)) \tag{1}$$

$\ddot{\mathbf{r}}_i$ can be recognized as the acceleration of the point mass i, so Newton's law is often memorized as "F=m a". Equation (1) is a *second order* ordinary differential equation, so when viewed as an initial value problem starting at time t_0, both $\mathbf{r}(t_0)$ and $\dot{\mathbf{r}}(t_0)$ are required to fully specify the motion at later times. The problem is mathematically well-posed when specified as follows:

$$\begin{cases} m_i \ddot{\mathbf{r}}_i(t) = \mathbf{F}_i(t, \mathbf{r}_1(t), \dots, \mathbf{r}_N(t), \dot{\mathbf{r}}_1(t) \dots \dot{\mathbf{r}}_N(t)) \qquad \text{for } t \geq t_0 \ , \ i = 1 \dots N \\ \mathbf{r}_i(t_0) = \mathbf{r}_{i0} \ , \quad i = 1 \dots N \\ \dot{\mathbf{r}}_i(t_0) = \mathbf{v}_{i0} \ , \quad i = 1 \dots N \end{cases} \tag{2}$$

Note the very remarkable fact – although by now quite intuitive for most human beings – that if the functional form of the forces is known for all times, then the motion of the point masses can be computed for

all times $t \geq t_0$. In fact, it can also be reconstructed for times $t \leq t_0$. The system is entirely deterministic. That does not always mean that the motion can be computed numerically with satisfying accuracy for all times. This is because the dynamics can be chaotic, and very small changes in the initial conditions can lead to significant differences at later times. This is the case for the motion of objects (planets, satellites, asteroids) in the solar system, or for the dynamics of a driven pendulum, a situation we will study in detail later in the course.

Let us now look at a few standard applications of Newton's law, to refresh our memory and gain familiarity with systems we will regularly go back to in the course of the semester.

1.2.2 Examples

- **Object in free fall under the effect of gravity: "Newton's apple"**

  The apple falling from the tree onto Newton's head can be idealized as a point mass with mass m and if the initial conditions are such that the apple does not have any horizontal velocity initially, the system has one degree of freedom, described by the vertical coordinate z. The apple is subject to the force of gravity, which physicists have shown to be equal to $\mathbf{F}_g = -mg\mathbf{e}_z$ if the z-axis is pointing upward, with $g \approx 9.8 m.s^{-2}$ the gravity constant. Newton's equation for the z coordinate is

$$m\ddot{z} = -mg \qquad \ddot{z} = -g \tag{3}$$

  This is the well-known result that in free fall, the magnitude of the acceleration is equal to the gravity constant. If we are given initial conditions $z(0) = z_0$ and $\dot{z}(0) = v_0$, Equation (3) is easily integrated, to give the motion for any later time t

$$z(t) = z_0 + v_0 t - \frac{g}{2}t^2$$

  Conservation of energy

  Multiplying Equation (3) by $\dot{z}$, we have

$$\dot{z}\ddot{z} + g\dot{z} = 0 \;\Leftrightarrow\; \frac{d}{dt}\left(\frac{\dot{z}^2}{2} + gz\right) = 0 \tag{4}$$

  We conclude that the quantity

$$H(z, \dot{z}) = \frac{\dot{z}^2}{2} + gz \tag{5}$$

  is conserved during the motion of the apple. The function $H(z, \dot{z})$ is called the *Hamiltonian*, and represents the energy of the system. We will go back to Hamiltonians very soon in this course. For the moment, it suffices to say that the first term in the Hamiltonian $\frac{\dot{z}^2}{2}$ is called the *kinetic energy*, and gz is the *potential energy*. Physicists usually define $m\frac{\dot{z}^2}{2}$ and mgz as the kinetic and potential energy respectively.

  Phase space

  As we have seen previously, the position and velocity of a point mass are sufficient to unequivocally specify its motion at later times, provided we know the force acting on it. In classical mechanics, the pair position-velocity is called the state of the system.

  A very efficient way to vizualize the dynamics of a system is to look at the physically allowed trajectories of the state of the system in the space of all imaginable states. This space is usually called *phase space* for historical reasons, and the plot of the allowable trajectories is often called a phase portrait.

  Since for Newton's apple $\frac{\dot{z}^2}{2} + gz$ is a constant quantity equal to the total energy of the apple, the phase portrait for Newton's apple is obtained by plotting the contours in the $(z, \dot{z})$ phase space of the Hamiltonian $H(z, \dot{z})$ for several values of the total energy. It is clear from the form of the Hamiltonian that these contours are parabolas, as shown in Figure 3.

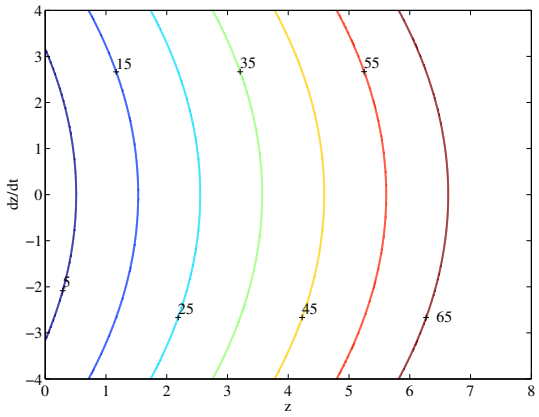

Figure 3: Phase portrait for Newton's apple ($g = 9.8$ was used for this figure)

- **The simple pendulum**

  The simple pendulum is a standard system in classical mechanics, made of a bob of mass m attached to a rigid rod of length l and negligible mass as compared to the mass of the bob, which is attached to a frictionless pivot (see Figure 4). One could use the x and y coordinates of the position of the bob to describe the dynamics, but it is not the most intuitive strategy. The reason is that the length l of the rod is fixed, so we have the constraint

  $$x^2 + y^2 = l^2$$

  In other words, the system has only one degree of freedom, and it is more convenient to use the coordinate θ to describe the dynamics of the pendulum. Let us therefore consider the natural polar coordinate system (r, θ) associated with the geometry, and fix $r = l$ to take the constraint into account.

  In polar coodinates, the position of an object is

  $$\mathbf{r} = r\mathbf{e}_r$$

  where $\mathbf{e}_r$ is the unit vector in the r direction. Since $\mathbf{e}_r$ depends on θ, the velocity of an object in polar coordinates is

  $$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

  where $\mathbf{e}_\theta$ is the unit vector in the θ direction, which also depends on θ. The acceleration is

  $$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

  In our case, $r = cst = l$, so the acceleration takes the simple form

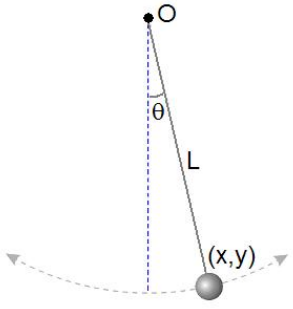

Figure 4: Schematic of the simple pendulum

$$\ddot{\mathbf{r}} = -l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta$$

There are two forces acting on the bob: the gravity force and the tension force of the rod on the bob. The tension force is entirely in the radial direction. The component of the gravity force in the θ direction is $F_\theta = -mg\sin\theta$ Therefore, Newton's law in the $\mathbf{e}_\theta$ direction is

$$ml\ddot{\theta} = -mg\sin\theta \qquad \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{6}$$

In the small angle limit, $\theta \to 0$, $\sin\theta \sim \theta$, so Equation (6) becomes the differential equation for a simple harmonic oscillator

$$\ddot{\theta} + \omega^2\theta = 0 \tag{7}$$

where

$$\omega = \sqrt{\frac{g}{l}}$$

is the angular frequency of the motion. The solution of Equation (7) is straightforward:

$$\theta(t) = \theta_0 \cos(\omega t + \phi)$$

where θ_0 is the maximum amplitude of the motion, and ϕ the phase.

Conservation of energy

Just as in the case of Newton's apple, we can derive a relation for the conservation of energy for the simple pendulum by multiplying Equation (6) by $\dot{\theta}$ and integrating

$$\frac{d}{dt}\left(\frac{\dot{\theta}^2}{2} - \frac{g}{l}\cos\theta\right) = 0 \quad \Leftrightarrow \quad H(\theta,\dot{\theta}) = \frac{\dot{\theta}^2}{2} - \frac{g}{l}\cos\theta = Cst = E \tag{8}$$

where $H(\theta,\dot{\theta})$ is the Hamiltonian for the simple pendulum. The first term in the Hamiltonian corresponds to the kinetic energy of the pendulum; the second term is the potential energy.

Phase space

The value of phase portraits becomes truly apparent in the case of the simple pendulum. It is quite easy, using a computer, to plot the contours of the Hamiltonian for the pendulum. This is what we do in Figure 5. We only show the portrait for $\theta \in [-\pi\ \pi]$ since the whole portrait is easily produced using the 2π periodicity of the equations. We recognize the key features of the simple pendulum, which would not be obtained so easily by looking at the differential equation (6):

- closed contours, corresponding to energies below the threshold for the full turn around the pivot; by taking the small angle limit for the Hamiltonian, we can see that these contours are ellipses
- the open contours corresponding to energies allowing the bob to fully turn around the pivot
- the yellow curve is the separatrix, corresponding to $E_{th} = g/l$.

- **Charged particles in a uniform magnetic field**

  We finish this section with the study of the motion of charged particles in a uniform magnetic field. The reason we look at this particular example, which is not often covered in classical mechanics courses, is that this situation leads to a more general form for the Lagrangian and for conjugate momenta than the forms usually presented in elementary presentations of Lagrangian mechanics. We will therefore discuss this situation in the following lectures, and this is a good place to introduce it.

  Consider space equipped with the Cartesian coordinates (x, y, z) and filled with a uniform magnetic field in the z-direction: $\mathbf{B} = B_0\mathbf{e_z}$. Physicists have discovered that a charged particle with charge q and velocity $\dot{\mathbf{r}}$ is subject to the Lorentz force

$$\mathbf{F}_L = q\dot{\mathbf{r}} \times \mathbf{B} \tag{9}$$

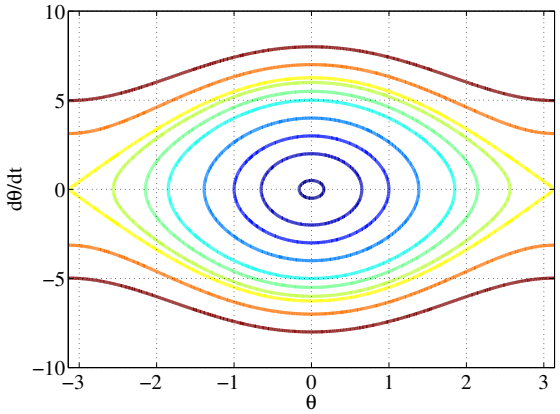

Figure 5: Phase portrait for the simple pendulum ($l = 1$ and $g = 9.8$ was used for this figure)

In the following, we will consider that the mass of the particle is small enough that gravity can be ignored. This is a very good approximation for electrons, protons and nuclei inside atoms for example. The system has three degrees of freedom, and given the geometry of the problem, the Cartesian coordinates (x, y, z) are appropriate.

Conservation of energy

We invert the order of the presentation and first derive the equation for the conservation of energy. The reason for this is that we will make explicit use of the conservation of kinetic energy when deriving the equations for the motion of the particle.

Newton's law for a particle of mass m subject to the Lorentz force is

$$m\frac{d^2\mathbf{r}}{dt^2} = q\dot{\mathbf{r}} \times \mathbf{B} \tag{10}$$

Dotting this equation with $\dot{\mathbf{r}}$, we immediately find

$$\frac{d}{dt}\left(m\frac{\dot{\mathbf{r}}^2}{2}\right) = 0$$

The Lorentz force does not do any work on moving particles, so the kinetic energy is conserved.

Motion of charged particles in a uniform magnetic field

Defining $\omega_c = qB_0/m$, called the cyclotron frequency, Newton's equations can be written as

$$\ddot{x} = \omega_c \dot{y} \tag{11}$$
$$\ddot{y} = -\omega_c \dot{x} \tag{12}$$
$$\ddot{z} = 0 \tag{13}$$

Equation (13) is readily solved:

$$\dot{z}(t) = \dot{z}(0) = v_{z0} \tag{14}$$

The motion in the direction of the magnetic field is unaffected by the magnetic force. The motion perpendicular to the magnetic field is more interesting. Taking a time derivative of Equation (11) and using (12), we find

$$\dddot{x} + \omega_c^2 \dot{x} = 0 \tag{15}$$

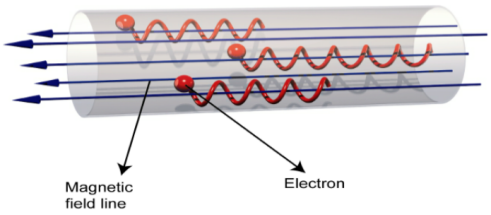

Figure 6: Helical trajectories of electrons immersed in a uniform magnetic field

The general solution of Equation (15) is

$$\dot{x}(t) = A\cos(\omega_c t + \varphi) \tag{16}$$

where the amplitude A and the phase φ are determined from the initial conditions. Plugging (16) into (11), we then have

$$\dot{y}(t) = -A\sin(\omega_c t + \varphi) \tag{17}$$

Now we saw earlier that the Lorentz force conserves the kinetic energy of the particle. The parallel kinetic energy is constant, as we saw in Equation (14). This means that the *perpendicular* kinetic energy, defined as

$$W_\perp = \frac{1}{2}mv_\perp^2 = \frac{1}{2}m\left(v_x^2 + v_y^2\right)$$

is also constant:

$$W_\perp = cst = \frac{1}{2}mv_{\perp 0}^2 \tag{18}$$

For any time t, we thus have

$$\dot{x}^2 + \dot{y}^2 = A^2 = v_{\perp 0}^2$$

so that the velocities are given for all times t by

$$\dot{x} = v_{\perp 0}\cos(\omega_c t + \varphi) \qquad \dot{y} = -v_{\perp 0}\sin(\omega_c t + \varphi) \tag{19}$$

These equations clearly describe a *circular* motion. Let us call (x_{gc}, y_{gc}) the center of that circle, called the *gyrocenter*. Integrating Equations (19) we obtain the particle's trajectory for the perpendicular motion:

$$x(t) = x_{gc} + \frac{v_{\perp 0}}{\omega_c}\sin(\omega_c t + \varphi) \qquad y(t) = y_{gc} + \frac{v_{\perp 0}}{\omega_c}\cos(\omega_c t + \varphi) \tag{20}$$

The radius of the circular motion is

$$\rho_L = \frac{v_{\perp 0}}{\omega_c} \tag{21}$$

and is called the *Larmor radius* or *gyroradius*.

$\omega_c > 0$ for positively charged particles, while $\omega_c < 0$ negatively charged particles. This means that positively charged particles and negatively charged particles rotate around the magnetic field in opposite directions. If the magnetic field is pointing toward you, positively charged particles rotate clockwise and negatively charged particles rotate counter-clockwise.

The trajectory in the parallel direction is found by integrating Equation (14): $z(t) = z_{gc} + v_{z0}t$ where z_{gc}, the z-coordinate of the guiding center position is also the z-coordinate z_0 of the initial position of the particle. Summarizing, the total trajectory is given by

$$x(t) = x_{gc} + \frac{v_{\perp 0}}{\omega_c}\sin(\omega_c t + \varphi) \tag{22}$$

$$y(t) = y_{gc} + \frac{v_{\perp 0}}{\omega_c}\cos(\omega_c t + \varphi) \tag{23}$$

$$z(t) = z_{gc} + v_{z0}t \tag{24}$$

Clearly, these are the parametric equations of a helical motion (see Figure 2).

1.3 Motivation for a variational formulation of classical mechanics

From the examples above, it seems like quite a lot can be done with the Newtonian approach to classical mechanics. In the case of the simple pendulum, we even managed to incorporate the constraint that the rod be rigid in deriving the equations of motion. Why then look at other formulations of classical mechanics? There are several reasons, that will become more and more apparent as the course unfolds. We will mention a few here:

- In the Newtonian approach, each point mass is treated individually. It is inherently a particle-by-particle approach. In contrast, as we will see, the variational formulation considers the various forms of energy in the system. These different energies do not depend on the way the system is described
- The equations of motion are derived in the same way regardless of the choice of the coordinate system
- In the variational formulation, constraints can be expressed in an easier manner, and even built into the coordinates
- The variational formulation provides a natural avenue for revealing symmetries and conserved quantities in the system
- The variational formulation can lead to significantly simpler derivations of the motion within the framework of asymptotic theories
- The variational formulation shares very close similarities with other fields of physics, such as quantum mechanics

2 Calculus of Variations

Before we go into the details of the variational formulations of classical mechanics, we will have a brief review of a fundamental result of the calculus of variation: the Euler-Lagrange equation. The presentation follows in a large part that of Oliver Bühler in *A Brief Introduction to Classical, Statistical, and Quantum Mechanics.*

2.1 Variation of functionals

Consider a function q of one variable defined on the interval $[a, b]$, and smooth enough to allow us to take all the derivatives we need to take.

We define the following integral

$$S[q] = \int_a^b L(q(x), \frac{dq}{dx}(x), x)dx \tag{25}$$

where L is sufficiently smooth in all its three variable slots to allow partial derivatives up to second order to exist. S is a *scalar* function that depends on the *function* q. We say that S is a *functional* of q, and the dependence of S on q is usually denoted with a square bracket. In the variational formulation of classical mechanics, we will extremize a functional called the action. Let us therefore look at the change of S as the function q is subject to a small variation. More precisely, we want to see how S changes as we let

$$q \qquad \rightarrow \qquad q + \delta q$$

where δq is small in the sense that

$$\sup\{|\delta q(x)|, x \in [a, b]\} \ll 1 \qquad \text{and} \qquad \sup\{\left|\frac{d}{dx}(\delta q(x))\right|, x \in [a, b]\} \ll 1$$

The variation of S can be approximated as follows

$$\begin{aligned} S[q + \delta q] - S[q] &= \int_a^b L(q + \delta q(x), \frac{dq}{dx} + \frac{d\delta q}{dx}, x)dx - \int_a^b L(q(x), \frac{dq}{dx}, x)dx \\ &= \int_a^b \left[\partial_1 L(q, \frac{dq}{dx}, x)\delta q + \partial_2 L(q, \frac{dq}{dx}, x)\frac{d\delta q}{dx}\right] dx + o(\delta q, \frac{d\delta q}{dx}) \end{aligned}$$

where the symbol ∂_i indicates a partial derivative with respect to whichever symbol appears in the i-th slot of L. The second term inside the square bracket can be integrated by part, so the integral in the equation above, called the first variation of S and written δS becomes

$$\delta S = \int_a^b \left[\partial_1 L(q, \frac{dq}{dx}, x) - \frac{d}{dx}\left(\partial_2 L(q, \frac{dq}{dx}, x)\right)\right] \delta q dx + \left[\partial_2 L(q, \frac{dq}{dx}, x)\delta q\right]_a^b \tag{26}$$

2.2 Extremals

The functional S is said to be extremized, and the function q is an extremal of S if δS measured around q vanishes for all δq. Looking at (26), this can only be the case if q solves the following second-order ordinary differential equation:

$$\frac{d}{dx}\left[\partial_2 L(q, \frac{dq}{dx}, x)\right] = \partial_1 L(q, \frac{dq}{dx}, x) \tag{27}$$

Indeed, if that were not true, then we could choose a function δq such that $\delta q(a) = \delta q(b) = 0$ and such that the integral is not zero. For such a function, we would have $\delta S \neq 0$, contradicting the fact that S is extremized.

Equation (27) is called the Euler-Lagrange equation, discovered jointly by Euler and Lagrange in the 18th century, as they were working on several problems in classical mechanics.

To see why the Euler-Lagrange equation lead to a second-order ODE in general, one can use the chain rule to rewrite the left-hand side of Equation (27):

$$\frac{d}{dx}\left[\partial_2 L(q, \frac{dq}{dx}, x)\right] = \partial_1\partial_2 L(q, \frac{dq}{dx}, x)\frac{dq}{dx} + \partial_2\partial_2 L(q, \frac{dq}{dx}, x)\frac{d^2q}{dx^2} + \partial_3\partial_2 L(q, \frac{dq}{dx}, x)$$

There are many possible boundary conditions on q, depending on the question one is asking. Often, $q(a)$ and $q(b)$ are fixed, as will be the most common case for us. These boundary conditions are called *rigid boundary conditions*, or *essential boundary conditions*. If $q(b)$ is not fixed, $\delta S = 0$ and Equation (26) imply that

$$\partial_2 L(q(b), \frac{dq}{dx}(b), b) = 0$$

This boundary condition is called a *natural boundary condition*.

2.3 Illustration: shortest path between two points

What we have just learned can be used to determine the shortest path between two points A and B with coordinates (x_A, y_A) and (x_B, y_B) in two-dimensional Euclidean space. The length functional is, assuming that $x_B > x_A$

$$S = \int_A^B ds = \int_A^B \sqrt{dx^2 + dy^2} = \int_{x_A}^{x_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

We can identify the integrand with $L(y, \frac{dy}{dx}, x) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ so that the function y that minimizes S satisfies the following Euler-Lagrange equation

$$\frac{d}{dx}\left(\partial_2 L\right) = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}} = cons$$

where $y' \equiv dy/dx$ and the boundary conditions are $y(x_A) = y_A$ and $y(x_B) = y_B$. The equation above implies that $y' = const$, so the extremal y is a straight line, as one would expect.

Note that in principle, we should do more work to prove that the extremal above corresponds to a *minimum* of S, and not a *maximum*. This is not always so easy; fortunately, it is clear that in our situation, we indeed found the minimum.

2.4 Euler-Lagrange equations for functionals of several functions

In the course of the semester, we will encounter systems that have more than one degree of freedom. To be ready for such situations, we need to consider functionals S that depend on N functions q_n. If the integrand has the following dependence, $L(q_1, \ldots, q_N, \frac{dq_1}{dx}, \ldots, \frac{dq_N}{dx}, x)$, you can repeat the steps we derived above for the integrand that only depended on one function, and convince yourself that the Euler-Lagrange equations determining the N functions extremizing the functional are

$$\frac{d}{dx}\left(\partial_{N+i} L\right) = \partial_i L \qquad i = 1, \ldots, N \tag{28}$$

This represents a system of N coupled second-order ODEs for the functions q_i.

Illustration: shortest path between two points

To illustrate the generalization above, let's reconsider the question of the shortest path between the points $A(x_A, y_A)$ and $B(x_B, y_B)$, viewing all possible paths $\gamma(t)$ between these two points as parametrized by the parameter t:

$$\gamma(t) : \{x(t), y(t)\} \ \ , t \in [0,1] \ , \ (x(0), y(0)) = (x_A, y_A) \ \ , \ \ (x(1), y(1)) = (x_B, y_B)$$

The length functional is

$$S = \int_A^B ds = \int_A^B \sqrt{dx^2 + dy^2} = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{29}$$

We can identify the functional

$$L(x, y, \frac{dx}{dt}, \frac{dy}{dt}, t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

so the Euler-Lagrange equations are

$$\begin{cases} \frac{d}{dt}\left(\partial_3 L\right) = 0 \ \ \Rightarrow \ \ \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = const \\ \frac{d}{dt}\left(\partial_4 L\right) = 0 \ \ \Rightarrow \ \ \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = const \end{cases} \tag{30}$$

where we have used the notation $\dot{x} = dx/dt$ and $\dot{y} = dy/dt$. The coupled equations (30) implies that there exists a constant C such that $\dot{y} = C\dot{x}$.γ thus is a straight line, as expected.