1 More general Lagrangians

1.1 Forces that depend on the velocity: an example

In Lecture 1, we have encountered a force that depended on the velocity: the Lorentz force. Such a force cannot be written as the gradient of a potential that only depends on time and positions. One can therefore not use Hamilton's principle to construct the Lagrangian. Let us see how one would proceed to construct the Lagrangian for the motion of a charged particle immersed in a static magnetic field.

We use Cartesian coordinates throughout, and start with the Lagrangian for the free particle: $1/2m|\dot{\mathbf{r}}|^2$. The Lagrangian must be a scalar quantity. The simplest scalar quantity that depends on the velocity $\dot{\mathbf{r}}$ and is not explicitly dependent on time that one can add to the Lagrangian for the free particle has the form

$$\dot{\mathbf{r}} \cdot \mathbf{W}(\mathbf{r})$$

where $\mathbf{W}$ is some vector field. Let us plug the Lagrangian

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m |\dot{\mathbf{r}}|^2 + \dot{\mathbf{r}} \cdot \mathbf{W}(\mathbf{r})$$

in the E-L equations and see if we can characterize $\mathbf{W}$ so L properly describes the dynamics of a particle in a static magnetic field.

For each component i, the $E - L$ equation is

$$\frac{d}{dt}\left(m\dot{r}_i + W_i\right) = \dot{\mathbf{r}} \cdot \frac{\partial \mathbf{W}}{\partial r_i}$$

Using the chain rule, this can be rewritten as

$$m\ddot{r}_i + \sum_j \frac{\partial W_i}{\partial r_j}\dot{r}_j = \sum_j \dot{r}_j \frac{\partial W_j}{\partial r_i}$$

$$\Leftrightarrow \; m\ddot{r}_i = \sum_j \dot{r}_j \left(\frac{\partial W_j}{\partial r_i} - \frac{\partial W_i}{\partial r_j}\right)$$

In the expression above, we recognize the components of the curl of $\mathbf{W}$, and we can rewrite the equation as

$$m\frac{d^2\mathbf{r}}{dt^2} = \dot{\mathbf{r}} \times (\nabla \times \mathbf{W}) \tag{1}$$

Equation (1) is Newton's law for a charged particle in a static magnetic field provided $q\mathbf{B} = \nabla \times \mathbf{W}$. In other words, we can identify $\mathbf{W}$ with the magnetic vector potential $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$: $\mathbf{W} = q\mathbf{A}$, and a Lagrangian for a charged particle in a static magnetic field is

$$L(\mathbf{r}, \frac{d\mathbf{r}}{dt}, t) = \frac{1}{2} m |\dot{\mathbf{r}}|^2 + q\dot{\mathbf{r}} \cdot \mathbf{A} \tag{2}$$

1.2 Systems with constraints

We have already encountered a system with constraints: the simple pendulum. The length of the rigid rod was fixed to be l. For that case, we managed to use a coordinate system that made it easy to us to impose the constraint. Specifically, by choosing polar coordinates (r, θ), we were able to impose $r = l$ and find an appropriate Lagrangian that naturally incorporated the rigid rod constraint. In many cases, however, it is much more complicated, and sometimes impossible, to find a coordinate transformation that naturally takes contraints into account. For these cases, the method of *Lagrange multipliers* can be used to transform the variational problem with constraints into a new unconstrained problem with more degrees of freedom. It works as follows.

1.2.1 Calculus of variation for constrained systems

Consider that the true trajectory in configuration space must satisfy a constraint of the form

$$f(\mathbf{q}, \dot{\mathbf{q}}, t) = 0$$

where $\mathbf{q} = (q_1, \ldots, q_N)$ and $\dot{\mathbf{q}} = (\dot{q}_1, \ldots, \dot{q}_N)$. The variational formulation of mechanics says that the physically realizable motion $\mathbf{q}$ is such that the action

$$S[\mathbf{q}] = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) dt$$

is extremized, where the endpoints $\mathbf{q}(t_1)$ and $\mathbf{q}(t_2)$ are fixed, and the functions $\mathbf{q}$ are taken in the space of functions that satisfy the constraint $f(\mathbf{q}, \dot{\mathbf{q}}, t) = 0$.

Taking the variation of S by letting $\mathbf{q} \to \mathbf{q} + \delta\mathbf{q}$, we obtain the sane condition for $\delta S = 0$ as we had previously obtained:

$$\int_{t_1}^{t_2} \left[\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \mathbf{q}} - \frac{d}{dt} \left(\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}} \right) \right] \cdot \delta\mathbf{q} = 0$$

This is the same condition as before, except that this time the functions $\delta\mathbf{q}$ must be such that the varied paths $\mathbf{q} + \delta\mathbf{q}$ also satisfy the constraint. In other words, the $\delta\mathbf{q}$ are not as arbitrary as in the unconstrained case.

We may still conclude that the following equality is true for all times:

$$\left[\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \mathbf{q}} - \frac{d}{dt} \left(\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}} \right) \right] \cdot \delta\mathbf{q} = 0 \tag{3}$$

This is true because from a $\delta\mathbf{q}$ that is such that the constraint is satisfied at all times, one can always construct another $\delta\tilde{\mathbf{q}}$ such that the constraints are satisfied and that is zero everywhere except for a very narrow time window where it is equal to $\delta\mathbf{q}$.

The difference with the unconstrained cases, however, is that one can no longer conclude that the term in the square brackets is equal to $\mathbf{0}$. All one can say is that on the physically realizable path,

$$\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \mathbf{q}} - \frac{d}{dt} \left(\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}} \right)$$

is orthogonal to any $\delta\mathbf{q}$ such that $f(\mathbf{q} + \delta\mathbf{q}, \dot{\mathbf{q}} + \delta\dot{\mathbf{q}}, t) = 0$. We will use this fact shortly, after characterizing these $\delta\mathbf{q}$ a bit better.

1.2.2 Holonomic constraints and augmented Lagrangian

Since the constraint $f(\mathbf{q}, \dot{\mathbf{q}}, t) = 0$ applies to all varied paths, $f(\mathbf{q} + \delta\mathbf{q}, \dot{\mathbf{q}} + \delta\dot{\mathbf{q}}, t) = 0$, the difference $f(\mathbf{q} + \delta\mathbf{q}, \dot{\mathbf{q}} + \delta\dot{\mathbf{q}}, t) - f(\mathbf{q}, \dot{\mathbf{q}}, t)$ must be 0 to all orders in $\delta\mathbf{q}$. In particular, we can write the following for the first order term:

$$\frac{\partial f}{\partial \mathbf{q}} \cdot \delta\mathbf{q} + \frac{\partial f}{\partial \dot{\mathbf{q}}} \cdot \delta\dot{\mathbf{q}} = 0 \tag{4}$$

We say that the variation $\delta\mathbf{q}$ is *tangent to the constraint surface* given by $f(\mathbf{q}, \dot{\mathbf{q}}, t) = 0$.

Let us first consider constraints that are independent of the generalized velocities $\dot{\mathbf{q}}$: $f(\mathbf{q}, t) = 0$. In that case, Equation (4) becomes

$$\frac{\partial f}{\partial \mathbf{q}} \cdot \delta\mathbf{q} = 0 \tag{5}$$

and an important relationship can be derived from Equations (3) and (4). Indeed, both $\left[\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \mathbf{q}} - \frac{d}{dt} \left(\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}} \right) \right]$ and $\frac{\partial f}{\partial \mathbf{q}}$ are orthogonal to any $\delta\mathbf{q}$ tangent to the constraint surface, so they must be colinear. There exists a scalar $\lambda(t)$ such that

$$\left[\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \mathbf{q}} - \frac{d}{dt} \left(\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}} \right) \right] = \lambda(t) \frac{\partial f}{\partial \mathbf{q}} \tag{6}$$

The scalar λ is called a *Lagrange multiplier*. In Equation (6), we recognize the EL equations associated with the coordinates $\mathbf{q}$ for the following Lagrangian

$$\tilde{L}(\mathbf{q}, \lambda, \dot{\mathbf{q}}, \dot{\lambda}, t) = L(\mathbf{q}, \dot{\mathbf{q}}, t) - \lambda(t) f(\mathbf{q}, t) \tag{7}$$

$\tilde{L}$ is called an *augmented Lagrangian*. The method developed above is called the method of Lagrange multipliers. It transforms a constrained variational problem into a new unconstrained problem with more degrees of freedom (as many as there are constraints, and hence λ). Once the expression for the augmented Lagrangian $\tilde{L}$ is obtained, one can apply the EL equations in the usual way, to all the degrees of freedom of $\tilde{L}$, and one obtains coupled ODEs describing the dynamics of the constrained problem.

The method of Lagrange multipliers also works if the constraint is $g = 0$ where g depends on the generalized velocities in such a way that g is the total time derivative of a velocity independent function $f(t, \mathbf{q})$:

$$g = \frac{d}{dt}f(t, \mathbf{q}) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} \tag{8}$$

Integrating Equation (8), one sees that $g = 0$ can also be written as $f = C$, where C is some constant. We can therefore apply what we have learned previously to write the following Euler-Lagrange equations for the system subject to the constraint $f(t, \mathbf{q}) - C = 0$

$$\left[\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \mathbf{q}} - \frac{d}{dt}\left(\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}}\right)\right] = \lambda(t)\frac{\partial f}{\partial \mathbf{q}}$$

So far, we have almost not done anything that is new. What is more interesting is that one can construct an augmented Lagrangian in terms of g directly instead of f. The logic for this is as follows. From Equation (8), one finds the following equality

$$\frac{\partial g}{\partial \dot{\mathbf{q}}} = \frac{\partial f}{\partial \mathbf{q}}$$

Plugging this result in the EL equations above, one can rewrite

$$\left[\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \mathbf{q}} - \frac{d}{dt}\left(\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}}\right)\right] = \lambda(t)\frac{\partial g}{\partial \dot{\mathbf{q}}} \tag{9}$$

One recognizes here the EL equations for the augmented Lagrangian

$$\tilde{L} = L + \tilde{\lambda} g$$

One can verify that the EL equations for $\tilde{L}$ are indeed in agreement with Equation (9), provided $d\tilde{\lambda}/dt = \lambda$.

Velocity constraints that can be written in terms of the derivative of coordinate-only constraints are called *integrable constraints*. Systems subject to constraints that are coordinate-only constraints or can be put in the form of coordinate-only constraints are called *holonomic systems*.

1.2.3 Example

Let us apply what we have just learned for the almost trivial case of the simple pendulum. Since the force on the bob is conservative, we know that a Lagrangian for the system is $L = T - V$

In polar coordinates, we have $T = \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right]$ and $V = -gr\cos\theta$. Furthermore, the constraint that the rod be rigid with length l can be written as $r = l$. An appropriate augmented Lagrangian for this system therefore is

$$\tilde{L}(r, \theta, \lambda, \dot{r}, \dot{\theta}, \dot{\lambda}, t) = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + gr\cos\theta - \lambda(r - l)$$

The EL equations for r, θ and λ are

$$\ddot{r} = r\dot{\theta}^2 + g\cos\theta - \lambda \tag{10}$$

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = -gr\sin\theta \tag{11}$$

$$r - l = 0 \tag{12}$$

The system above is particularly easy to solve. Equation (12) is the constraint, as one would expect. It gives a value for r that can be used in the other two equations. Inserting $r = l$ into (11), we recover the second order ODE for θ for the simple pendulum, which we have already seen twice. Equation (10) becomes

$$\lambda = l\dot{\theta}^2 + g\cos\theta$$

which gives an expression for λ once $\theta(t)$ is known. One can show that λ is equal to the force along the rod of the pendulum that is necessary to enforce the constraint. This is a general result of the method of Lagrange multipliers applied to Lagrangian mechanics: the Lagrange multipliers are proportional to the forces required to enforce the constraints.

1.2.4 Nonholonomic constraints

Systems with constraints that are not integrable are called *nonholonomic systems*. You may find in the literature Euler-Lagrange-like equations for systems with such constraints, and attempts to show that these equations can be derived from the action principle. None of the derivations I know of are convincing and appear rigorous. In the context of this class, we will therefore stick to situations for which we have proven rigorous results, namely situations with holonomic constraints.

2 Coordinate transformation

One of the strengths of the Lagrangian formulation of mechanics is that the generic form of the EL equations is independent of the choice of coordinates used to express the Lagrangian. Specifically, consider a Lagrangian $L(q, \frac{dq}{dt}, t)$ and q that satisfies the EL equation

$$\frac{d}{dt}\left(\partial_2 L\right) = \partial_1 L \tag{13}$$

Now, consider the coordinate transformation $q = f(Q, t)$ and the transformed Lagrangian

$$\tilde{L}(Q, \frac{dQ}{dt}, t) = L(f(Q,t), \partial_1 f(Q,t)\frac{dQ}{dt} + \partial_2 f(Q,t), t)$$

We have

$$\begin{aligned}\partial_1 \tilde{L}(Q, \frac{dQ}{dt}, t) &= \partial_1 L \partial_1 f + \partial_2 L \left[\partial_1 \partial_1 f \frac{dQ}{dt} + \partial_1 \partial_2 f(Q,t)\right] \\ \partial_2 \tilde{L}(Q, \frac{dQ}{dt}, t) &= \partial_2 L \partial_1 f(Q,t)\end{aligned}$$

Therefore

$$\frac{d}{dt}\left(\partial_2 \tilde{L}\right) = \frac{d}{dt}\left(\partial_2 L\right)\partial_1 f(Q,t) + \partial_2 L \partial_1 \partial_1 f \frac{dQ}{dt} + \partial_2 L \partial_2 \partial_1 f(Q,t) = \partial_1 L \partial_1 f + \partial_2 L\left(\partial_1 \partial_1 f \frac{dQ}{dt} + \partial_1 \partial_2 f(Q,t)\right) \tag{14}$$

$$= \partial_1 \tilde{L} \tag{15}$$

where we have used Equation (13) and the fact that $\partial_1 \partial_2 f = \partial_2 \partial_1 f$ for smooth enough coordinate transformations.

We have just proven that Q and the transformed Lagrangian $\tilde{L}$ satisfy an EL equation that is formally identical to the EL equation satisfied by q and L.

3 Conserved quantities

3.1 Conservation of momentum

Consider a Lagrangian $L(q_1, \ldots, q_N, \frac{dq_1}{dt}, \ldots, \frac{dq_N}{dt}, t)$ that does not depend explicitly on the coordinate q_i. Then the E-L equation for the coordinate q_i is

$$\frac{d}{dt}\left(\partial_{N+i} L\right) = 0$$

We conclude that the quantity

$$p_i = \partial_{N+i} L = \frac{\partial L}{\partial \dot{q}_i} \tag{16}$$

is conserved. The second expression in Equation 16 is used much more often than the first expression, is more explicit, but perhaps also more dangerous. The conserved quantity p_i is called a *generalized momentum*, and is also often called the momentum *conjugate* to the coordinate q_i.

Going back to the example of the object of mass m orbiting around the much heavier object, we saw that the Lagrangian did not explicitly depend on the angle θ. This means that

$$p_\theta = \partial_4 L(r, \theta, \frac{dr}{dt}, \frac{d\theta}{dt}, t) = mr^2 \frac{d\theta}{dt}$$

is a conserved quantity, as we have seen. In this case, p_θ, the momentum conjugate to the variable θ, is the angular momentum.

3.2 Conservation of energy

If the Lagrangian of a system does not depend explicitly on time, then the energy of the system is conserved. This can be shown as follows. We consider the Lagrangian $L(q_1, \ldots, q_N, \frac{dq_1}{dt}, \ldots, \frac{dq_N}{dt})$ and calculate

$$\frac{dL}{dt} = \sum_{i=1}^{N} \left(\partial_i L \frac{dq_i}{dt} + \partial_{i+N} L \frac{d^2 q_i}{dt^2} \right) = \sum_{i=1}^{N} \left(\partial_i L \frac{dq_i}{dt} + p_i \frac{d^2 q_i}{dt^2} \right)$$

Along a physically realizable path, L satisfies the E-L equations, so that

$$\partial_i L = \frac{d}{dt} \left(\partial_{i+N} L \right) = \frac{dp_i}{dt}$$

We thus have

$$\frac{dL}{dt} = \sum_{i=1}^{N} \left(\frac{dp_i}{dt} \frac{dq_i}{dt} + p_i \frac{d^2 q_i}{dt^2} \right) = \frac{d}{dt} \left(\sum_{i=1}^{N} p_i \frac{dq_i}{dt} \right)$$

We conclude that

$$\frac{d}{dt} \left(\sum_{i=1}^{N} p_i \frac{dq_i}{dt} - L \right) = 0 \tag{17}$$

$\sum p_i \dot{q}_i - L$ is a conserved quantity called the *energy* of the system.

Illustration for a charged particle immersed in a static magnetic field

We saw that a Lagrangian for such a system is $L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m |\dot{\mathbf{r}}|^2 + q \dot{\mathbf{r}} \cdot \mathbf{A}$. The conjugate momentum is

$$\mathbf{p} = \nabla_{\dot{\mathbf{r}}} L = m \dot{\mathbf{r}} + q \mathbf{A} \tag{18}$$

Note that the expression is more general than the formula $\mathbf{p} = m\mathbf{v}$ one tends to be used to. $\mathbf{p}$ is the generalized momentum, whereas $m\mathbf{v}$ is the linear momentum. The energy of the charged particle, which is conserved since L does not depend explicitly on t, is

$$E = \mathbf{p} \cdot \dot{\mathbf{r}} - L = \frac{1}{2} m |\dot{\mathbf{r}}|^2$$

in agreement we what we have seen in the last lecture.

3.3 Noether's theorem

The situations we studied in the previous two subsections are all particular cases of a general result first obtained by Emmy Noether, and which can be formulated as follows: for each symmetry of the Lagrangian, there is a conserved quantity. By symmetry, we mean here a coordinate transformation that leaves the Lagrangian unchanged. Here is a proof of this theorem:

Consider a Lagrangian $L(q, \frac{dq}{dt}, t)$ and a parametric coordinate transformation

$$q = M(s)(Q, t)$$

You can think of M as a rotation for example, and s would then be the angle, as we will see in the next subsection. The state $(q, dq/dt, t)$ of the system is transformed as $(q, dq/dt, t) = (M(s)(Q, t), \frac{d}{dt} \left[M(s)(Q, t) \right], t)$. We require that the transformation $M(0)$ is the identity transformation.

Consider then the transformed Lagrangian

$$\tilde{L}(s)(Q, dQ/dt, t) = L(M(s)(Q, t), \frac{d}{dt} \left[M(s)(Q, t) \right], t)$$

The Lagrangian L has a continuous symmetry corresponding to the transformation M if $\tilde{L}(s) = L$ *for any* s. This means that $d\tilde{L}/ds = 0$:

$$\begin{aligned} 0 &= \partial_1 L(q, dq/dt, t)\frac{dM}{ds}(Q,t) + \partial_2 L(q, dq/dt, t)\frac{d}{ds}\left[\frac{d}{dt}\left[M(s)(Q,t)\right]\right] \\ &= \partial_1 L(q, dq/dt, t)\frac{dM}{ds}(Q,t) + \partial_2 L(q, dq/dt, t)\frac{d}{dt}\left[\frac{d}{ds}\left[M(s)(Q,t)\right]\right] \end{aligned}$$

Now, along a physically realizable path, L satisfies the E-L equations:

$$\frac{d}{dt}(\partial_2 L(q, dq/dt, t)) = \partial_1 L(q, dq/dt, t)$$

Using this in the first term on the right-hand side of the previous expression, we can write

$$\begin{aligned} &\frac{d}{dt}\left[\partial_2 L(q, dq/dt, t)\right]\frac{dM}{ds}(Q,t) + \partial_2 L(q, dq/dt, t)\frac{d}{dt}\left[\frac{d}{ds}\left[M(s)(Q,t)\right]\right] = 0 \\ \Leftrightarrow\ &\frac{d}{dt}\left[\partial_2 L(q, dq/dt, t)\frac{dM}{ds}(Q,t)\right] = 0 \end{aligned}$$

When $s = 0$, $(q, dq/dt, t) = (Q, dQ/dt, t)$, and the equality above implies that

$$(\partial_2 L)\left.\frac{dM}{ds}\right|_{s=0} \tag{19}$$

evaluated at $(q, dq/dt, t)$ on the physical path is conserved. This conserved quantity is called *Noether's integral*

Note that Noether's theorem is even more general than our proof suggests: it also applies to transformations M that depend on dQ/dt.

3.4 Illustration of Noether's theorem

In the previous lecture, we looked at the case of a point mass in a central force potential $V(|\mathbf{r}|)$. Let us return to that case in the context of Noether's theorem.

We write the Lagrangian in Cartesian coordinates:

$$L(x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, t) = \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] - V(\sqrt{x^2+y^2+z^2})$$

and we consider a parametric rotation $R_z(s)$ of the coordinate system about the z axis:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_z(s)\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x'\cos s - y'\sin s \\ x'\sin s + y'\cos s \\ z' \end{pmatrix} \tag{20}$$

The rotation preserves lengths since it is an orthogonal transformation, so

$$x^2 + y^2 + z^2 = (x')^2 + (y')^2 + (z')^2$$

Furthermore, differentiating (20) along a path in configuration space gives

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = R_z(s)\begin{pmatrix} \frac{dx'}{dt} \\ \frac{dy'}{dt} \\ \frac{dz'}{dt} \end{pmatrix}$$

so the velocities also satisfy

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2$$

Therefore,

$$\tilde{L}(x', y', z', \frac{dx'}{dt}, \frac{dy'}{dt}, \frac{dz'}{dt}, t) = \frac{1}{2}m\left[\left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2\right] - V(\sqrt{x'^2 + y'^2 + z'^2})$$
$$= \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] - V(\sqrt{x^2 + y^2 + z^2}) = L(x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, t)$$

The Lagrangian is invariant under the rotation of the coordinate system around the z axis. According to Noether's theorem, the quantity $\partial_2 L(\mathbf{r}, \frac{d\mathbf{r}}{dt}, t)\frac{dR_z}{ds}(0)$ is conserved.

$$\partial_2 L(\mathbf{r}, \frac{d\mathbf{r}}{dt}, t) = m\frac{d\mathbf{r}}{dt} = m\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$\left.\frac{dR_z(s)}{ds}\right|_{s=0}(x, y, z) = (-y, x, 0)$$

Hence, Noether's integral is

$$m(x\frac{dy}{dt} - y\frac{dx}{dt})$$

We recognize here the z component of the angular momentum $\mathbf{r} \times (m d\mathbf{r}/dt)$. It is conserved for the central potential problem. In fact, there is nothing special about the z-axis here, and one can easily show in a similar way that all components of the angular momentum are conserved.

4 A first step towards quantum mechanics

4.1 Maupertuis' principle: Spatial form for the principle of stationary action

Consider a particle with mass m and electric charge q subject to external forces such that a Lagrangian describing its motion is

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - V(\mathbf{r}) + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r})$$

One can imagine modifying the principle of stationary action in such a way that we only consider variations of the action corresponding to motions with a given, fixed total energy E.

For the Lagrangian considered here, we have

$$E = \frac{\partial L}{\partial \dot{\mathbf{r}}} \cdot \mathbf{r} - L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + V(\mathbf{r})$$

The action we want to extremize is

$$S = \int_{t_1}^{t_2} L(\mathbf{r}, \dot{\mathbf{r}}, t)dt = \int_{t_1}^{t_2} \left(\frac{1}{2}m|\dot{\mathbf{r}}|^2 - V(\mathbf{r}) + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r})\right)$$
$$= \int_{t_1}^{t_2} \left(m|\dot{\mathbf{r}}|^2 + q\dot{\mathbf{r}} \cdot \mathbf{A}\right) dt - \int_{t_1}^{t_2} E dt$$

If we only consider variations of S for motions with a fixed energy E, the second term is $-E(t_2 - t_1)$. In that case, extremizing S is equivalent to extremizing Maupertuis' action S_M, given by

$$S_M = \int_{t_1}^{t_2} \left(m|\dot{\mathbf{r}}|^2 + q\dot{\mathbf{r}} \cdot \mathbf{A}\right) dt = \int_{t_1}^{t_2} (m\dot{\mathbf{r}} + q\mathbf{A}) \cdot \dot{\mathbf{r}} dt \tag{21}$$

We recognize the generalized momentum $\mathbf{p}$ in the integrand, so we can write

$$S_M = \int_{t_1}^{t_2} \mathbf{p} \cdot \dot{\mathbf{r}} dt$$

We can now use the change of variable $\mathbf{dl} = \dot{\mathbf{r}}dt$ to eliminate the time and transform the integral above into an integral over the spatial trajectory between the points M_1 and M_2:

$$S_M = \int_{M_1}^{M_2} \mathbf{p} \cdot \mathbf{dl} \tag{22}$$

This is the original form of the principle of stationary action, as introduced by Maupertuis in 1744. The action to extremize is the circulation of the vector $\mathbf{p}$ between the points M_1 and M_2.

Equation (22) reveals an interesting analogy with geometric optics, which we will highlight in the next section.

4.2 From Fermat's principle to Maupertuis' principle

As we already saw in Homework 1, in the approximation of geometric optics, the trajectory of light in a medium with refraction index $n(\mathbf{r})$ is such that the optical path S_F is extremized, where S_F is given by

$$S_F = \int_{M_1}^{M_2} n(\mathbf{r}) dl \tag{23}$$

This principle was first formulated by Fermat. The refractive index is defined by

$$n(\mathbf{r}) = \frac{c}{v_{phase}(\mathbf{r})}$$

where $v_{phase} = \omega/k$ is the phase velocity of light in the medium, $\omega = 2\pi\nu$ where ν is the frequency of the light ray, and k is the magnitude of the wave vector $\mathbf{k}$. In other words, extremizing S_F as given in equation (23) is equivalent to extremizing the following action, which is dimensionless:

$$\tilde{S}_F = \int_{M_1}^{M_2} \mathbf{k} \cdot \mathbf{dl} \tag{24}$$

The analogy between Equations (22) and (24) could only make sense with the advent of quantum mechanics, and the famous formula first proposed by Louis De Broglie associating to each particle with momentum $\mathbf{p}$ a wave whose wave vector $\mathbf{k}$ is such that

$$\mathbf{p} = \hbar \mathbf{k}$$

where $\hbar = h/2\pi$ and h is the Planck constant, with units of $J.s$. Comparing (22) and (24), we see that $\hbar$ is the natural unit of action.