1 Motion of a free rigid body

1.1 Lagrangian for the free rigid body

Applying Hamilton's principle, it is clear that the Lagrangian of a free body is just the kinetic energy of that body. By choosing variables in a clever way, we managed to write the kinetic energy in terms of E_C, which depends only on the velocity of the center of mass, and in terms of E_R, which depends only on the rotational motion of the body around its center of mass. We thus see that the full Lagrangian for the free rigid body decouples into two independent parts: a Lagrangian L_C associated with the uniform motion of the center of mass, and a Lagrangian L_R associated with the rotational motion. Here, we concentrate on the interesting part of the motion, namely the rotational motion with Lagrangian L_R.

We have

$$L_R(\theta,\phi,\psi,\dot\theta,\dot\phi,\dot\psi,t) = \frac{1}{2}\left[I_{11}\left(\dot\phi\sin\psi\sin\theta+\dot\theta\cos\psi\right)^2 + I_{22}(\dot\phi\sin\theta\cos\psi-\dot\theta\sin\psi)^2 + I_{33}(\dot\phi\cos\theta+\dot\psi)^2\right] \tag{1}$$

In principle, all we need to fully determine the dynamics of the body is in the Lagrangian. We could write down the E-L equations associated with it, and solve the ordinary differential equations for θ, ϕ, and ψ. While it is an unpleasant calculation to have to do by hand, it is very easy to do with a computer. Even if so, a fully numerical solution may not provide all the insights regarding the motion of the body one may get from having a close look at key properties of the system that are fairly natural to derive by hand. This is precisely what we do now.

1.2 Conserved quantities

The first observation is that L_R does not depend on time explicitly. Therefore, the energy is conserved by the motion. Second, we also notice that L_R does not depend explicitly on ϕ, so the conjugate momentum p_ϕ is conserved.

We find

$$\begin{aligned} p_\phi = \frac{\partial L_R}{\partial\dot\phi} &= I_{11}\left(\sin^2\psi\sin^2\theta\dot\phi+\cos\psi\sin\psi\sin\theta\dot\theta\right) \\ &+ I_{22}\left(\sin^2\theta\cos^2\psi\dot\phi-\sin\theta\cos\psi\sin\psi\dot\theta\right) \\ &+ I_{33}\left(\cos^2\theta\dot\phi+\cos\theta\dot\psi\right) \end{aligned}$$

The reason we wrote this horrible expression explicitly is that it is equal to L_z, the z component of the angular momentum. Indeed, we know that the component $\mathbf{L}'$ of the angular momentum along the principal axes are

$$\begin{aligned} \mathbf{L}' &= (I_{11}\Omega_1, I_{22}\Omega_2, I_{33}\Omega_3) \\ &= (I_{11}(\dot\phi\sin\psi\sin\theta+\dot\theta\cos\psi), I_{22}(\dot\phi\sin\theta\cos\psi-\dot\theta\sin\psi), I_{33}(\dot\phi\cos\theta+\dot\psi)) \end{aligned}$$

Therefore, the components $\mathbf{L}$ of the angular momentum in the $(\hat x,\hat y,\hat z)$ basis are

$$\mathbf{L} = \mathbf{R}\mathbf{L}'$$

If you were to calculate $\mathbf{L}$ as above, you would find that the third component of this matrix equation, L_z is equal to p_ϕ. Since p_ϕ is conserved, L_z is conserved as well. Now, since the orientation of the basis is completely arbitrary, we could have chosen a different orientation and found that another rectangular component of $\mathbf{L}$ is conserved. This means that $\mathbf{L}$ itself is conserved.

The spatial components of the angular momentum are conserved, but the projections of the angular momentum onto the principal axes change because these axes move. Let us look at the evolution of these projections. There are two things we know: the kinetic energy of the rigid body is conserved, and its angular momentum is conserved. Combining Equations (10) and (14) from Lecture 4, the conservation of kinetic energy can be written as

$$E_R = \frac{1}{2}\left(\frac{L_1^2}{I_{11}}+\frac{L_2^2}{I_{22}}+\frac{L_3^2}{I_{33}}\right) = Cst \tag{2}$$

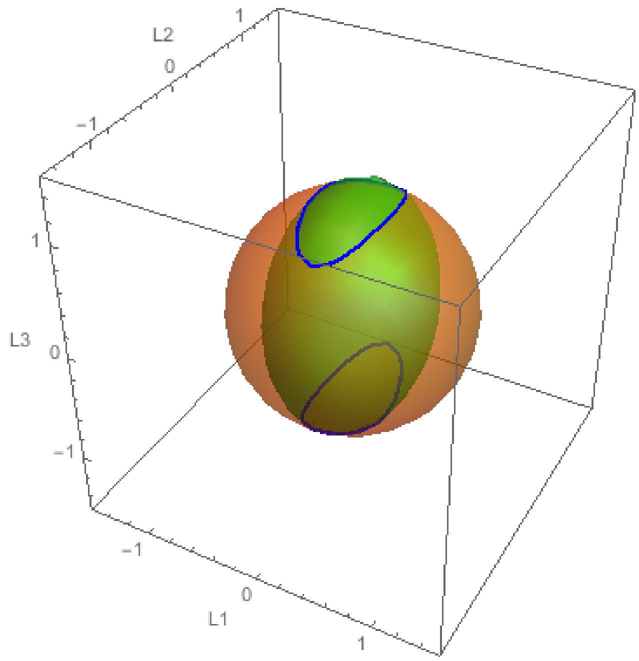

Figure 1: Energy ellipsoid in green and angular momentum sphere in orange. The two curves of intersection are highlighted in blue; the components of the angular momentum on the principal axes are constrained to follow these curves in their evolution.

The conservation of the magnitude of the angular momentum can be written as

$$L^2 = L_1^2 + L_2^2 + L_3^2 = Cst \tag{3}$$

In Equation (2), we recognize the equation of an ellipsoid, and in (3) the equation of a sphere. Both are centered in $(0, 0, 0)$. The components of the angular momentum along the principal axes are constrained to evolve on the intersection of these two surfaces. This intersection is in general two closed curves, so we conclude that the components of the angular momentum along the principal axes are constrained to follow one of these curves. This is shown in Figure 1, in which we plot the energy ellipsoid in green and the angular momentum sphere in orange

Let us assume, without loss of generality, that $I_{33} > I_{22} > I_{11}$, and look at what happens when we fix the magnitude of the total angular momentum L and vary E_R. There is only a range of admissible energies. The minimum energy is such that $E_R = \frac{L^2}{2I_{33}}$. This case is illustrated in Figure 2. We see something remarkable: the sphere and the ellipsoids are tangent at only two points. In other words, for this value of the energy, all the angular momentum is along the third principal axis and remains there. We can view these points as equilibria for the components of the angular momentum. They are sometimes called *relative equilibria* since the body itself is rotating and is therefore not in equilibrium.

The maximum energy is such that $E_R = \frac{L^2}{2I_{11}}$. This case is illustrated in Figure 3. We see a situation very similar to the previous case: the sphere and the ellipsoids are tangent at only two points. For this value of the energy, all the angular momentum is along the first principal axis, and remains there: these points are two more relative equilibria.

Before we go to the intermediate energy case, $E_R = \frac{L^2}{2I_{22}}$, let us look at what the intersection curves look like if we choose an energy slightly above the minimum energy or slightly below the maximum energy. These two situations are shown in Figure 4.

We see that intersection curves starting at any point in the neighborhood of the relative equilibria discussed so far remain in the neighborhood of these equilibria. We can say that these relative equilibria are *stable*.

In contrast, consider the intermediate energy case: $E_R = \frac{L^2}{2I_{22}}$, shown in Figure 5. The curves of intersection appear to intersect at two points that belong to the second principal axis, with associated moment I_{22}. In fact, it can be showed that the trajectories of the dynamical system giving the components of the angular momentum as a function of time cannot intersect. This is because the system of ordinary differential equations for the components (see the subsection on Euler's equations below) satisfies a Lipschitz condition

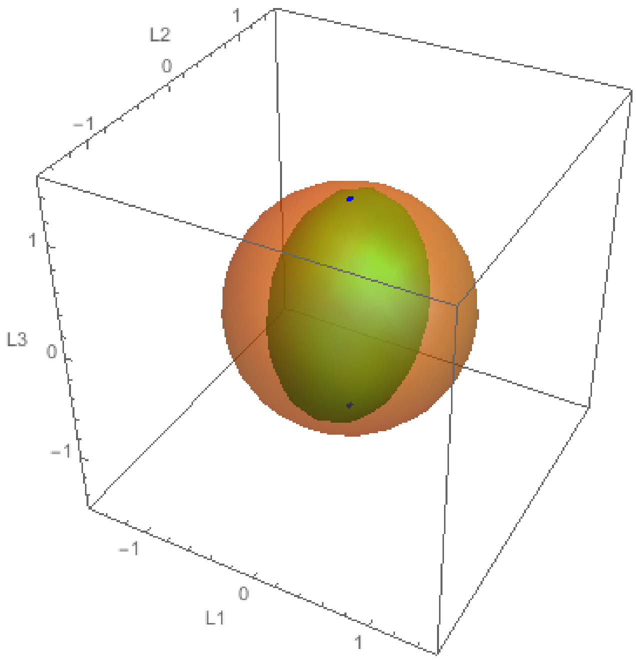

Figure 2: Energy ellipsoid in green and angular momentum sphere in orange for the minimum energy case: $E_R = \frac{L^2}{2I_{33}}$. The two curves of intersection are just two points

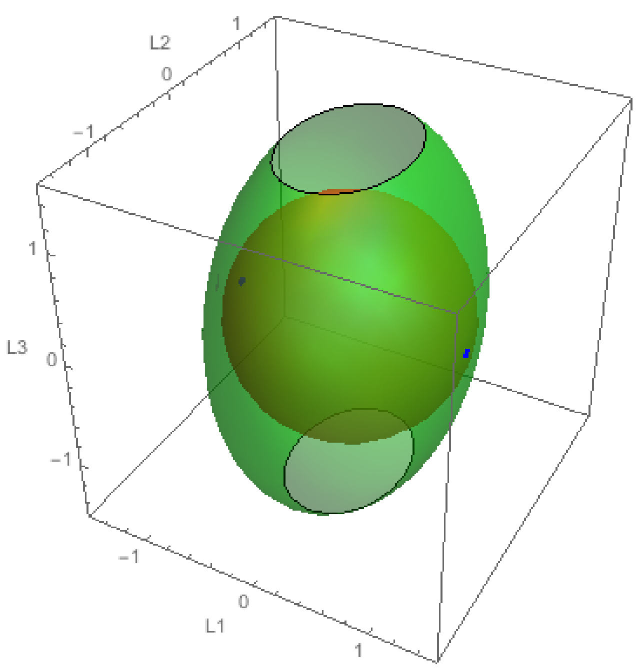

Figure 3: Energy ellipsoid in green and angular momentum sphere in orange for the maximum energy case: $E_R = \frac{L^2}{2I_{11}}$. The two curves of intersection are just two points

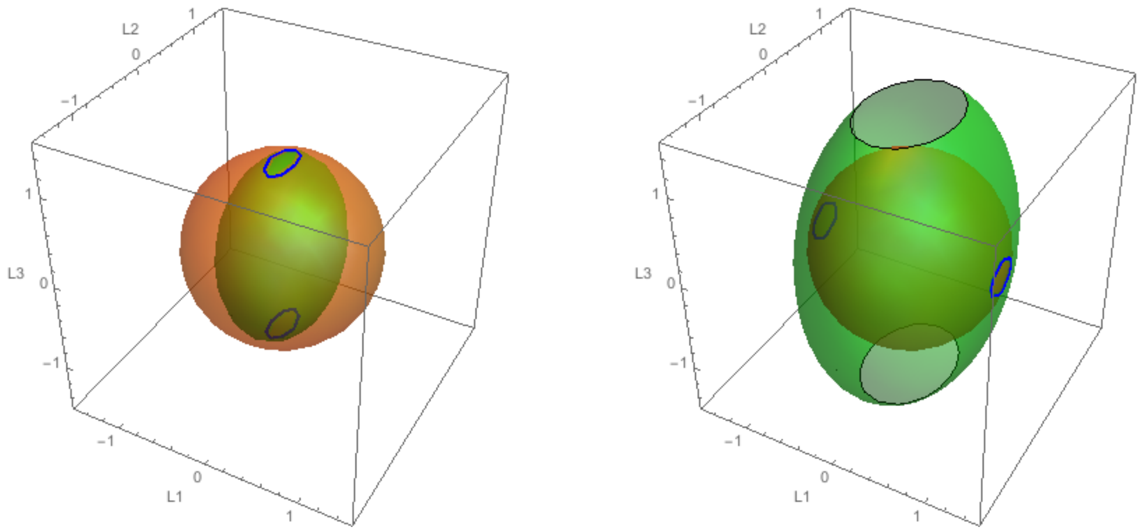

Figure 4: Curves of intersection between the energy ellipsoid (in green) and the angular momentum sphere (in orange) for an energy slightly above the minimum energy (left) and an energy slightly below the maximum energy (right)

and therefore has unique solutions. In other words, the only way to reach the separatrix is by starting there. Choosing points close to the separatrix as initial conditions, one can come arbitrarily close to the separatrix, but not reach it. The separatrix is an isolated equilibrium point for the components of the angular momentum, but an *unstable* one: any small displacement of the angular momentum leads to a large departure from the equilibrium situation.

In conclusion, we have found an explanation for our observations regarding the motion of a textbook thrown in the air: we found stable rotations when we chose to rotate the book about the axes with the smallest and largest moment of inertia, but could only get a tumbling motion when trying to rotate the book about the third axis. In the following subsection, we will give a slightly more quantitative explanation of the same phenomenon.

1.3 Euler's equations

We have seen that while the components of the angular momentum on the fixed rectangular basis are conserved quantities of the motion, the components on the principal axes are not. Let us derive equations for the evolution of these components.

Let $\mathbf{I}'$ be the diagonal inertia tensor, and $\mathbf{\Omega}'$ the components of the angular velocity vector on the principal axes, so that the components of the angular momentum vector on the principal axes is given by

$$\mathbf{L}' = \mathbf{I}'\mathbf{\Omega}'$$

Since the components $\mathbf{L}$ of the angular momentum vector on the rectangular basis are constant, we have

$$\frac{d}{dt}\mathbf{L} = \mathbf{0}$$

Now, we know that $\mathbf{L} = \mathbf{R}\mathbf{L}'$, so we can write

$$\frac{d\mathbf{R}}{dt}\mathbf{L}' + \mathbf{R}\frac{d\mathbf{L}'}{dt} = 0$$

This matrix equation is easily inverted to obtain an expression for the evolution of $\mathbf{L}'$ in time:

$$\frac{d\mathbf{L}'}{dt} = -\mathbf{R}^T\frac{d\mathbf{R}}{dt}\mathbf{L}'$$

This is a remarkable result: for a free rigid body, the variation of the components of the angular momentum on the principal axes only depends on these components themselves.

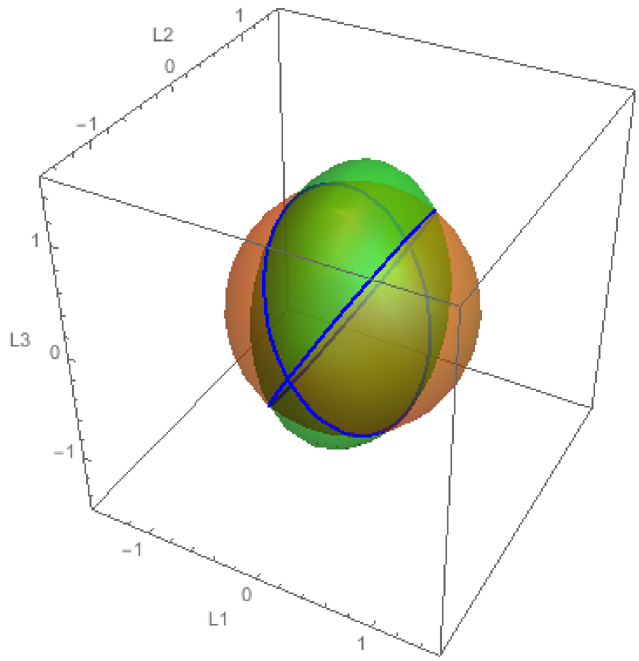

Figure 5: Energy ellipsoid in green and angular momentum sphere in orange for the intermediate energy case: $E_R = \frac{L^2}{2I_{22}}$. The two curves of intersection are just form a separatrix.

Let us see what this means for the evolution of $\mathbf{\Omega}'$:

$$\mathbf{I}'\frac{d\mathbf{\Omega}'}{dt} = -\mathbf{R}^T\frac{d\mathbf{R}}{dt}\mathbf{I}'\mathbf{\Omega}' \tag{4}$$

Now, remember that we had defined the matrix operator $\mathbf{C}$ such that

$$\mathbf{\Omega}^{'} = \mathbf{R}^T\mathbf{C}^{-1}\left(\frac{d\mathbf{R}}{dt}\mathbf{R}^T\right) \quad \Leftrightarrow \quad \mathbf{C}(\mathbf{R}\mathbf{\Omega}')\mathbf{R} = \frac{d\mathbf{R}}{dt}$$

Using this in Equation (4), we can write

$$\mathbf{I}'\frac{d\mathbf{\Omega}'}{dt} = -\mathbf{R}^T\mathbf{C}(\mathbf{R}\mathbf{\Omega}')\mathbf{R}\mathbf{I}'\mathbf{\Omega}'$$

Now, if R is the rotation and $\mathbf{u}$ and $\mathbf{v}$ are two vectors, we know that $R(\mathbf{u}\times\mathbf{v}) = R\mathbf{u}\times R\mathbf{v}$. We can use this fact to write the identity $\mathbf{R}^T\mathbf{C}(\mathbf{R}\mathbf{\Omega}^{'})\mathbf{R} = \mathbf{C}(\mathbf{\Omega}^{'})$, and thus finally find the desired relationship:

$$\mathbf{I}'\frac{d\mathbf{\Omega}'}{dt} = -\mathbf{C}(\mathbf{\Omega}')\mathbf{I}'\mathbf{\Omega}'$$

This is the matrix form of *Euler's equations*, giving the time derivative of the components of the angular velocity on the principal axes solely in terms of these components and the principal moments of inertia.

If the diagonal entries of $\mathbf{I}'$ are I_{11}, I_{22}, and I_{33} and if the components of $\mathbf{\Omega}'$ are $(\Omega_1, \Omega_2, \Omega_3)$, then Euler's equations are

$$\begin{aligned} I_{11}\frac{d\Omega_1}{dt} &= (I_{22} - I_{33})\Omega_2\Omega_3 \\ I_{22}\frac{d\Omega_2}{dt} &= (I_{33} - I_{11})\Omega_1\Omega_3 \\ I_{33}\frac{d\Omega_3}{dt} &= (I_{11} - I_{22})\Omega_1\Omega_2 \end{aligned} \tag{5}$$

Let us now see if we can use linear perturbation theory to explain the stability of the motion of the book thrown in the air.

Let us assume that at first, the rotation is completely around the first axis: $\mathbf{\Omega'} = (\Omega_1, 0, 0)$, and let us perturb this state by adding small rotations ϵ and μ in the other directions: $\mathbf{\Omega'} = (\Omega_1, \epsilon, \mu)$. Euler's equations become

$$\begin{aligned} I_{11}\frac{d\Omega_1}{dt} &= (I_{22} - I_{33})\epsilon\mu \\ I_{22}\frac{d\epsilon}{dt} &= (I_{33} - I_{11})\Omega_1\mu \\ I_{33}\frac{d\mu}{dt} &= (I_{11} - I_{22})\Omega_1\epsilon \end{aligned} \tag{6}$$

If we only look at the first order, linear behavior, the term $\epsilon\mu$ can be neglected, so to first order Ω_1 is a constant of the motion. The last two equations of the system can be rewritten as

$$\begin{aligned} \frac{d\epsilon}{dt} &= \frac{(I_{33} - I_{11})\Omega_1}{I_{22}}\mu \\ \frac{d\mu}{dt} &= \frac{(I_{11} - I_{22})\Omega_1}{I_{33}}\epsilon \end{aligned}$$

By differentiating one of the equations with respect to time, one of the variables is easily eliminated in terms of the other, and one finds that ϵ satisfies the following equation

$$\frac{d^2\epsilon}{dt^2} + \frac{(I_{22} - I_{11})(I_{33} - I_{11})}{I_{22}I_{33}}\Omega_1^2\epsilon = 0 \tag{7}$$

It is easy to show that μ satisfies the same equation.

Suppose now that $I_{11} < I_{22} < I_{33}$. In that case,

$$\Lambda^2 = \frac{(I_{22} - I_{11})(I_{33} - I_{11})}{I_{22}I_{33}}\Omega_1^2 > 0$$

and the motion described by Equation (7) corresponds to stable periodic motion. The body oscillates sinusoidally about its initial state.

Now imagine we repeat the analysis assuming a perturbation about the rotation $\mathbf{\Omega}' = (0, 0, \Omega_3)$. All the steps are identical to the ones above, except that we would find

$$\Lambda^2 = \frac{(I_{22} - I_{33})(I_{11} - I_{33})}{I_{11}I_{22}}\Omega_3^2 > 0$$

Since $\Lambda^2 > 0$, we find stable oscillations about its initial state in this case too.

Consider finally a perturbation around $\mathbf{\Omega}' = (0, \Omega_2, 0)$. Repeating the analysis, we would find

$$\Lambda^2 = \frac{(I_{33} - I_{22})(I_{11} - I_{22})}{I_{11}I_{33}}\Omega_2^2 < 0$$

Since $\Lambda^2 < 0$, the solution to Equation (7) is the sum of an exponentially growing function and exponentially decaying function. In other words, the small perturbations ϵ and μ grow exponentially (as long as the linear approximation is valid): the system is unstable.

2 Axisymmetric top

2.1 Lagrangian for the axisymmetric top

We now look at the axisymmetric top, whose geometry will allow us to proceed with a slightly more quantitative study than in the previous section. We consider an ideal top with total mass M, whose point of contact with the ground is fixed in space, and subject to gravitational forces, as shown in Figure 6. We have not yet dealt with potential energy in our study of rigid bodies. Let us therefore start by addressing this point.

Newtonian gravity forces are linear, so the total potential energy of the top is the sum of the potential energies of all the point masses in the top: $V = \sum_i m_i g h_i$, where h_i is the height of the point mass i above

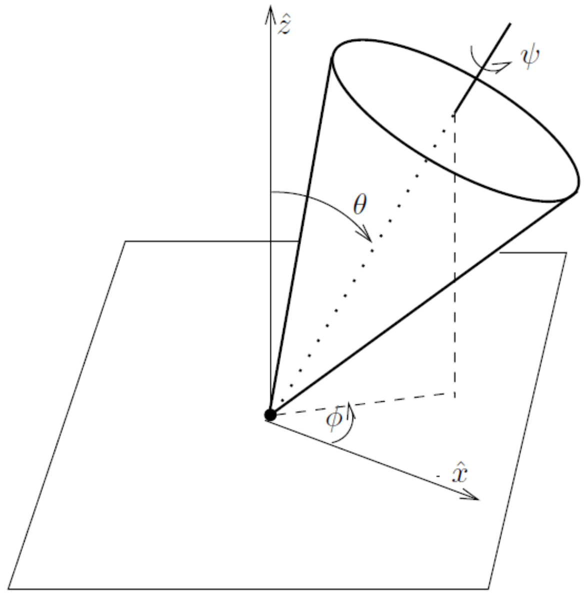

Figure 6: Sketch of the axisymmetric top with its associated Euler angles. Image reproduced from the textbook *Structure and Interpretation of Classical Mechanics*, MIT Press (2007), with permission of the authors. This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License. To view a copy of this license, visit creativecommons.org.

the ground. The idea is to try to write V in terms of the position of the center of mass $\mathbf{R}$ and the relative positions of the point masses $\mathbf{x}_i$. We have

$$\begin{aligned} V &= \sum_i m_i g h_i = \sum_i m_i g \mathbf{r}_i \cdot \hat{z} \\ &= Mg\mathbf{R} \cdot \hat{z} + g\hat{z} \cdot \sum_i m_i \mathbf{x}_i = Mg\mathbf{R} \cdot \hat{z} \end{aligned}$$

This is a very convenient result: the potential energy of the top is just Mgh, where h is the height of the center of mass.

The axisymmetric top appears to have 2 symmetries: 1) it is invariant under rotation about the line going through the pivot and its center of gravity; 2) because the gravitational potential is uniform, it is invariant under rotation around the z-axis. The first symmetry implies that the inertia tensor is such that $I_{11} = I_{22} \equiv I \neq I_{33}$. The two symmetries suggest that Euler angles may be a good choice to study the dynamics of the object. Let us construct a Lagrangian using these generalized coordinates.

Combining Equations (10) and (18) from Lecture 4, the kinetic energy is

$$\begin{aligned} E_R(\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}, t) &= \frac{I}{2}(\dot{\phi} \sin\psi \sin\theta + \dot{\theta}\cos\psi)^2 + \frac{I}{2}(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)^2 + \frac{I_{33}}{2}(\dot{\phi}\cos\theta + \dot{\psi})^2 \\ &= \frac{I}{2}(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{I_{33}}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 \end{aligned}$$

If R is the distance from the pivot to the center of mass, then the potential energy is

$$V(\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}, t) = mgR\cos\theta$$

We conclude, using Hamilton's principle, that a Lagrangian for the system is

$$L(\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}, t) = \frac{I}{2}(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{I_{33}}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - mgR\cos\theta \tag{8}$$

As always, one option at this point is to stop thinking: feed the Lagrangian into a software that can handle symbolic calculations to obtain E-L equations, and then numerically solve the ordinary differential equations. A smarter approach here is to use conserved quantities to reduce the problem to a simple one dimensional problem. This is what we will do in the following sections.

2.2 Conserved quantities

We first observe that the Lagrangian does not depend on time explicitly, so energy is conserved. We also see that neither ϕ nor ψ appear explicitly in the Lagrangian, so the conjugate momenta p_ϕ and p_ψ are conserved. They are

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I \sin^2\theta \dot{\phi} + I_{33} \cos\theta \left(\dot{\phi} \cos\theta + \dot{\psi} \right) \tag{9}$$

and

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_{33}(\dot{\phi} \cos\theta + \dot{\psi}) \tag{10}$$

The energy is

$$E = p_\theta \dot{\theta} + p_\phi \dot{\phi} + p_\psi \dot{\psi} - L$$

It turns out that the energy can be expressed in terms of θ, $\dot{\theta}$, and constants of the motion only. Let us first compute

$$p_\theta \dot{\theta} = \frac{\partial L}{\partial \dot{\theta}} \dot{\theta} = I \dot{\theta}^2$$

Now, combining Equations (9) and (10), we can write

$$p_\phi = I \sin^2\theta \dot{\phi} + \cos\theta p_\psi \quad \Leftrightarrow \quad \dot{\phi} = \frac{p_\phi - \cos\theta p_\psi}{I \sin^2\theta}$$

Furthermore, Equation (10) can be rewritten as

$$\dot{\psi} = \frac{p_\psi}{I_{33}} - \dot{\phi} \cos\theta$$

Hence,

$$\begin{aligned} E &= I\dot{\theta}^2 + p_\phi \frac{(p_\phi - \cos\theta p_\psi)}{I \sin^2\theta} + p_\psi \left(\frac{p_\psi}{I_{33}} - \frac{p_\phi - \cos\theta p_\psi}{I \sin^2\theta} \cos\theta \right) - \frac{I}{2} \sin^2\theta \left(\frac{p_\phi - \cos\theta p_\psi}{I \sin^2\theta} \right)^2 - \frac{I}{2}\dot{\theta}^2 - \frac{p_\psi^2}{2I_{33}} + mgR\cos\theta \\ &= \frac{I}{2}\dot{\theta}^2 + \frac{(p_\phi - \cos\theta p_\psi)^2}{I \sin^2\theta} + \frac{p_\psi^2}{2I_{33}} - \frac{(p_\phi - \cos\theta p_\psi)^2}{2I \sin^2\theta} + mgR\cos\theta \end{aligned}$$

$$E = \frac{I}{2}\dot{\theta}^2 + \frac{(p_\phi - \cos\theta p_\psi)^2}{2I \sin^2\theta} + \frac{p_\psi^2}{2I_{33}} + mgR\cos\theta \tag{11}$$

2.3 Qualitative description of the motion

Equation (11) can be seen as the following differential equation for θ:

$$\dot{\theta}^2 = \frac{2}{I}\hat{E} - \frac{(p_\phi - \cos\theta p_\psi)^2}{I^2 \sin^2\theta} - \frac{2mgR}{I}\cos\theta \tag{12}$$

where we have defined $\hat{E} = E - p_\psi^2/(2I_{33})$. This differential equation can be analysed more easily by letting $u = \cos\theta$. Equation (12) becomes

$$\dot{u}^2 = \frac{2}{I}\left(\hat{E} - mgRu\right)\left(1 - u^2\right) - \frac{(p_\phi - up_\psi)^2}{I^2} \equiv F(u)$$

In principle, one could integrate this equation. This is not a very pleasant thing to do. Fortunately, much can be learned about the system without much algebra. The equation

$$\dot{u}^2 = F(u)$$

only has physical solutions for $F(u) \geq 0$. Let us therefore look at the behavior of F to understand the limits on the evolution of θ. In general, $\theta \in [0, \pi]$. Given the presence of the ground, $0 \leq \theta \leq \pi/2$, so $0 \leq u \leq 1$.

F is a cubic polynomial in u, which has at most three real roots. It is easy to see that

$$\lim_{u \to -\infty} F(u) = -\infty \qquad \lim_{u \to +\infty} F(u) = +\infty$$

Furthermore, $F(-1) < 0$ and $F(1) < 0$, so F has at most two roots u_1 and u_2 in the interval $[-1, 1]$ (there is then a solution $u_3 > 1$). We conclude that in the interval $[-1, 1]$, F is positive between u_1 and u_2.

Let us first assume that both u_1 and u_2 are positive. Then, the corresponding values θ_1 and θ_2 are the bounds on θ. The top oscillates between these two values, in a motion that is called *nutation*. If $u_1 < 0$, the nutation motion causes the top to hit the ground. If there is a double real root: $u_1 = u_2 > 0$, there is no nutation motion, and the top precesses steadily.

2.4 Quantitative description of the motion

To conclude this section, we study a real top and solve the differential equations numerically. We do this in two steps. First, we explicitly write the E-L equation for the θ variable:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{\partial L}{\partial \theta} \\ \Leftrightarrow I\ddot{\theta} &= I\dot{\phi}^2 \sin\theta\cos\theta - I_{33}\dot{\phi}\sin\theta(\dot{\psi} + \dot{\phi}\cos\theta) + MgR\sin\theta \end{aligned} \tag{13}$$

The equation above does not only involve θ and time derivatives of θ, but also $\dot{\phi}$ and $\dot{\psi}$. However, we have previously derived expressions for $\dot{\phi}$ and $\dot{\psi}$ in terms of conserved quantities and θ only. Using them in Equation (13), we find, after a bit of algebra,

$$\ddot{\theta} = \frac{p_\phi - \cos\theta p_\psi}{I^2 \sin^3\theta}\left[\cos\theta\,(p_\phi - \cos\theta p_\psi) - p_\phi \sin^2\theta\right] + \frac{MgR}{I}\sin\theta \tag{14}$$

We now have a differential equation that only involves θ and constants of the motion, whose value can be obtained from the initial conditions.

Once θ is known, we can integrate the expression

$$\dot{\phi} = \frac{p_\phi - \cos\theta p_\psi}{I\sin^2\theta}$$

and obtain ϕ as a function of time. We can also compute $\dot{\psi}$ through

$$\dot{\psi} = \frac{p_\psi}{I_{33}} - \dot{\phi}\cos\theta$$

As a numerical example, we start with the same parameters as the ones considered in the textbook by Sussman and Wisdom. The top is such that $I = 3.28 \times 10^{-4}$ kg m^2, $I_{33} = 6.6 \times 10^{-5}$ kg m^2, $gMR = 0.0456$ kg m^2 s^{-2}. The initial conditions are $\dot{\psi} = 140$ radians s^{-1}, $\dot{\theta} = 0$ radians s^{-1}, $\phi = \psi = 0$ radians, and $\theta = 0.1$ radians. Furthermore, the top is kicked so that $\dot{\phi} = 140$ radians s^{-1}. The results are shown in Figure 7 below

We see that the tilt of the top, measured by the angle θ, changes in a periodic manner: this is the nutation motion we discussed above. We also see that the top rotates around the vertical axis, as measured by the angle ϕ. This is called the *precession* motion of the top. Note that the rate of precession is not steady, and there are even moments at which the precession angle decreases. To better visualize what is happening, consider the plot of θ versus ϕ at the bottom right: it appears that for the initial conditions we chose. the top in fact executes a looping motion. If we had not given a kick to the top ($\dot{\phi} = 0$), there would not be any looping motion: the loops in the plot would become cusps. If we had kicked the top in the other direction, we would have found neither cusps nor looping motion. It is a good exercise to verify this on your own!

A word of caution

In the derivation of the Lagrangian for the axisymmetric top, we used two critical shortcuts. First, the fixed point in the system is the pivot point, and not the center of mass. So the moments of inertia we should use to express the kinetic energy should be the moments of inertia with respect to the pivot point instead of the center of mass. Fortunately, it turns out that if we know the moments of inertia about the center of mass, we can easily write the moments of inertia with respect to the pivot in terms of them, thanks to a theorem called Steiner's theorem.

Second, we claimed that the kinetic energy of the top is the kinetic energy of rotation about the pivot point, without proving it.

In Homework # 3, you will show why the claim above is justified, and also prove Steiner's theorem, thereby giving us a warmer, more confident feeling regarding our study of the top.

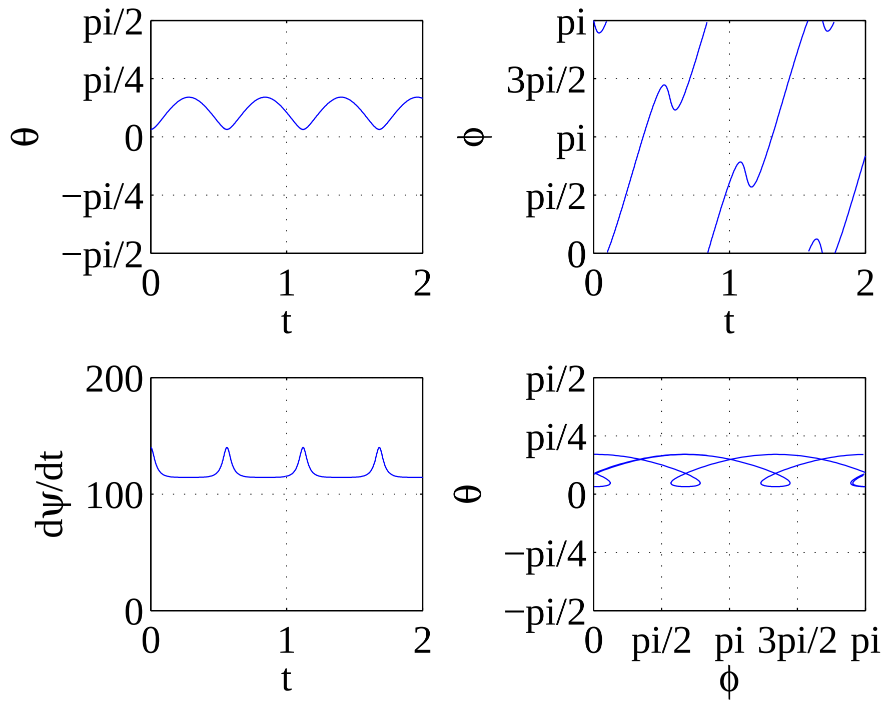

Figure 7: Dynamics of the looping axisymmetric top, obtained with $I = 3.28 \times 10^{-4}$ kg m^2, $I_{33} = 6.6 \times 10^{-5}$ kg m^2, $gMR = 0.0456$ kg m^2 s^{-2}, $\dot{\phi} = 140$ radians s^{-1}, $\dot{\psi} = 140$ radians s^{-1}, $\dot{\theta} = 0$ radians s^{-1}, $\phi = \psi = 0$ radians, and $\theta = 0.1$ radians.

3 Spin-Orbit coupling

3.1 Approximation for the potential energy

Our goal in the whole section is to understand key features of the dynamics of a rigid body in the gravitational potential of an external point mass. This is a very good idealization for the Sun-Mercury system, or the Earth-Moon system for example.

In the previous section, a key element that allowed us to proceed with a fairly simple analysis of the top was that the potential energy took a particularly simple form. In general, for the problem we are interested in here, there is no reason for the potential energy to take a simple form. However

Let M' be the mass of the distant point mass, m_i the masses of all the point masses in the rigid body, and $\mathbf{r}_i$ the vector between the location of the point mass i and the location of the external point mass M'. From the linearity of the gravitational force, the gravitational potential energy of the rigid body is

$$V = -\mathcal{G}M' \sum_i \frac{m_i}{|\mathbf{r}_i|}$$

As we did for the kinetic energy, we start by writing the $\mathbf{r}_i$ as the sum $\mathbf{r}_i = \mathbf{R} + \mathbf{x}_i$ of the location of the center of mass and the relative position $\mathbf{x}_i$ of all the point masses with respect to the center of mass.

We can write

$$V = -\mathcal{G}M' \sum_i \frac{m_i}{|\mathbf{R} + \mathbf{x}_i|}$$

To make further progress, we will consider the situation in which the distance from the rigid body to the external point mass M is much larger than the characteristic size of the rigid body. This is a very good approximations for the systems we have in mind (Earth-Moon, Sun-Mercury). We can then write a multipole expansion for the potential energy in the small ratio $|\mathbf{x}_i|/|\mathbf{R}| = d_i/R$, where we have defined $d_i = |\mathbf{x}_i|$:

$$V = -\mathcal{G}M' \sum_i \frac{m_i}{|\mathbf{R}+\mathbf{x}_i|} = -\mathcal{G}M' \sum_i \frac{m_i}{[(\mathbf{R}+\mathbf{x}_i)\cdot(\mathbf{R}+\mathbf{x}_i)]^{1/2}} = -\mathcal{G}M' \sum_i \frac{m_i}{[R^2 + 2\mathbf{R}\cdot\mathbf{x}_i + d_i^2]^{1/2}}$$
$$= -\mathcal{G}\frac{M'}{R} \sum_i \frac{m_i}{\left[1 + \frac{2}{R^2}\mathbf{R}\cdot\mathbf{x}_i + \left(\frac{d_i}{R}\right)^2\right]^{1/2}}$$
$$= -\mathcal{G}\frac{M'}{R} \sum_i \frac{m_i}{\left[1 - \frac{2d_i}{R}\cos\gamma_i + \left(\frac{d_i}{R}\right)^2\right]^{1/2}}$$

In all the above, γ_i is the angle between $\mathbf{x}_i$ and $-\mathbf{R}$. At this point, one recognizes the form of the generating function for Legendre polynomials, so the multipole expansion takes the following form

$$V = -\mathcal{G}\frac{M'}{R}\sum_i m_i \sum_{k=0}^{\infty} P_k(\cos\gamma_i)\left(\frac{d_i}{R}\right)^k = -\mathcal{G}\frac{M'}{R}\sum_{k=0}^{\infty}\sum_i m_i P_k(\cos\gamma_i)\left(\frac{d_i}{R}\right)^k$$

where P_k is the k-th Legendre polynomial. Since the Legendre polynomials all have magnitudes less than 1 for inputs in the range $[-1,1]$, it is easy to see that for $d_i/R \ll 1$, the terms in the multipole expansion decay relatively fast. We will only keep the first three terms in the multipole expansion. For $k = 0$, we have $P_0 = 1$, so the first term in the multipole expansion is

$$V = -\mathcal{G}\frac{M'M}{R}$$

where M is the total mass of the rigid body. This term is called the *monopole* term. The second term, corresponding to $k = 1$ is called the *dipole* term, and it is identically zero. This is because $P_1(\cos\gamma_i) = \cos\gamma_i$, and

$$\sum_i m_i d_i \cos\gamma_i = -\frac{1}{R}\sum_i m_i \mathbf{x}_i \cdot \mathbf{R} = -\frac{1}{R}\mathbf{R}\cdot\sum_i m_i\mathbf{x}_i$$

The third term in the expansion is the *quadrupole* term, corresponding to $k = 2$. We have $P_k(\cos\gamma_i) = 3/2\cos^2\gamma_i - 1/2$, so the quadrupole term is

$$-\mathcal{G}\frac{M'}{R^3}\sum_i m_i d_i^2\left(\frac{3}{2}\cos^2\gamma_i - \frac{1}{2}\right) = -\mathcal{G}\frac{M'}{R^3}\sum_i m_i d_i^2\left(1 - \frac{3}{2}\sin^2\gamma_i\right)$$

Now, remember that the moment of inertia I about a given line is given by

$$I = \sum_i m_i (d_i^\perp)^2$$

where $d_i^\perp$ is the distance from the point mass m_i to the axis. Therefore, the moment of inertia I_G of the rigid body about the axis alined with $\mathbf{R}$ is

$$I_G = \sum_i m_i (d_i \sin\gamma_i)^2 = \sum_i m_i d_i^2 \sin^2\gamma_i$$

We also know that there exists axes such that the inertia tensor $\mathbf{I}$ of the rigid body is diagonal, with entries I_{11}, I_{22}, and I_{33}. If in that basis, the components of $\mathbf{x}_i$ are (x_i', y_i', z_i'), then we can write

$$I_{11} = \sum_i m_i(y_i'^2 + z_i'^2) \qquad I_{22} = \sum_i m_i(x_i'^2 + z_i'^2) \qquad I_{33} = \sum_i m_i(x_i'^2 + y_i'^2)$$

It follows that

$$I_{11} + I_{22} + I_{33} = 2\sum_i (x_i'^2 + y_i'^2 + z_i'^2) = 2\sum_i d_i^2$$

We conclude that the quadrupole term takes the form

$$-\mathcal{G}\frac{M'}{2R^3}(I_{11} + I_{22} + I_{33} - 3I_G) \tag{15}$$

The last step consists in expressing I_G in terms of I_{11}, I_{22}, and I_{33}. If we call θ_a, θ_b and θ_c the angles between the principal axis $\hat{a}$, $\hat{b}$ and $\hat{c}$ of the rigid body and the axis aligned with $\mathbf{R}$, we can show that

$$I_G = \cos^2\theta_a I_{11} + \cos^2\theta_b I_{22} + \cos^2\theta_c I_{33}$$

so our approximation for the potential energy will be

$$V = -\mathcal{G}\frac{M'M}{R} - \mathcal{G}\frac{M'}{2R^3}\left[(1-3\cos^2\theta_a)I_{11} + (1-3\cos^2\theta_b)I_{22} + (1-3\cos^2\theta_c)I_{33}\right] \tag{16}$$

3.2 Reduced model for the spin-orbit coupling problem

We see from our approximate expression for the potential energy that it depends both on the position of the rigid body with respect to the external point mass and on the orientation of the rigid body. In general, we can therefore say that the change in the orientation is coupled to the orbital evolution. However, it turns out that in practice, the effect of the orientation of the body on the orbital evolution is very small. There are two reasons for this. First, it can be seen from our multipole expansion that the orientation of the rigid body and the departure of its shape from a purely spherical shape are $(d_{max}/R)^2$ smaller than the first term in the potential energy. Second, the shape of many of the objects we may study (the Moon, Mercury, ...) is very close to spherical, so that $(I_{11} + I_{22} + I_{33} - 3I_G)$ is a small quantity.

It remains interesting to see how the orientation of the rigid body evolves along a given orbit defined by the first term in the potential energy. This is what we do now, with the following simplified model, based on three assumptions

1. The rigid body is rotating about its largest moment of inertia
2. The axis of this rotation about itself ("spin") is perpendicular to the orbital motion
3. The orbit of the rigid body moves on a fixed elliptic orbit, with the external point mass at a focus of the ellipse

All these assumptions are justified for the study of the Earth-Moon system or many other objects in the Solar system, for reasons we will not discuss here.

The geometry of the problem is shown in Figure 8. The angle f measures the position of the rigid body along its orbit around the external mass, with the point at which R is smallest taken as reference point. Since we have assumed that the axis of spin is always perpendicular to the orbit, only one parameter is necessary to describe the orientation of the system. We choose it to be the angle θ shown in Figure 8, that measures the angle from the line used to measure f to the principal axis $\hat{a}$. θ will be the generalized coordinate used to write the Lagrangian.

Now, since we have assumed that the rigid body is rotating about its largest moment of inertia, the kinetic energy of rotation takes a particularly simple form:

$$E_R(\theta,\dot{\theta},t) = \frac{1}{2}I_{33}\dot{\theta}^2$$

After a little bit of elementary trigonometry, we find

$$\theta_a = \pi - (\theta - f) \qquad \theta_b = \frac{\pi}{2} - (\theta - f)$$

Therefore, the potential energy of the rigid body can be written as

$$V = -\mathcal{G}\frac{M'M}{R} - \mathcal{G}\frac{M'}{2R^3}\left[(1-3\cos^2(\theta-f))I_{11} + (1-3\sin^2(\theta-f))I_{22} + I_{33}\right]$$

Now, remember that we consider that the orbit of the rigid body is given to us. So all we have to keep in the Lagrangian are the terms in V associated with the orientation of the rigid body, i.e. which depend on θ and which we call V_R. After a little bit of algebra, we find that V_R can be written as

$$V_R(\theta,\dot{\theta},t) = -\frac{3}{4}\mathcal{G}\frac{M'}{R(t)^3}\,(I_{22} - I_{11})\cos[2(\theta - f(t))]$$

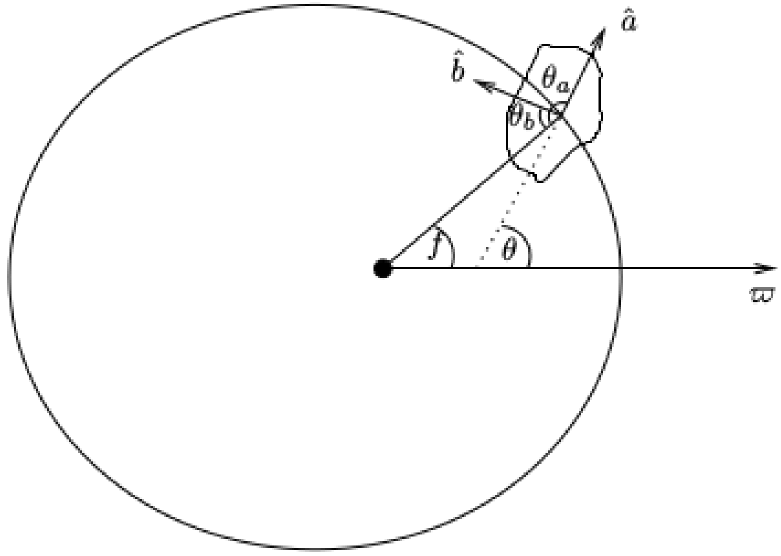

Figure 8: Geometry for the reduced model used to study spin-orbit coupling. Image reproduced from the textbook *Structure and Interpretation of Classical Mechanics*, MIT Press (2007), with permission of the authors. This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License. To view a copy of this license, visit creativecommons.org.

We conclude that the Lagrangian for the reduced model we use to study the spin-orbit coupling problem is

$$L(\theta, \dot{\theta}, t) = \frac{1}{2} I_{33} \dot{\theta}^2 + \frac{3}{4} \mathcal{G} \frac{M'}{R(t)^3} (I_{22} - I_{11}) \cos[2(\theta - f(t))]$$

We now introduce two dimensionless parameters that play a key role on the dynamics. The first parameter is the orbital period T. Kepler's third law tells us that T is related to a, the semi-major axis of the ellipse, through the following relationship:

$$\mathcal{G}(M + M') = 4\pi^2 \frac{a^3}{T^2}$$

Assuming $M' \gg M$, $\mathcal{G}M' = 4\pi^2 a^3/T^2$.

The second parameter is

$$\epsilon = \pi \sqrt{\frac{3(I_{22} - I_{11})}{I_{33}}}$$

The Lagrangian can be rewritten as follows in terms of these parameters

$$L(\theta, \dot{\theta}, t) = \frac{1}{2} I_{33} \dot{\theta}^2 + I_{33} \frac{\epsilon^2}{T^2} \frac{a^3}{R^3(t)} \cos[2(\theta - f(t))] \tag{17}$$

The E-L equation for the Lagrangian is easily found

$$\ddot{\theta} = -2 \frac{\epsilon^2}{T^2} \frac{a^3}{R^3(t)} \sin[2(\theta - f(t))] \tag{18}$$

Kepler's first law tell us that the orbit is described by an ellipse with the attracting mass M' as one of the foci:

$$R(t) = \frac{b^2}{a(1 + e\cos(f(t)))}$$

where b is the semi-minor axis of the ellipse, and e its eccentricity, such that $b = a\sqrt{1 - e^2}$.

Kepler's second law tells us that

$$R^2(t) \dot{f}(t) = \frac{2\pi}{T} ab$$

Consequently, $f(t)$ evolves according to

$$\frac{df}{dt} = \frac{2\pi}{T}(1-e^2)^{1/2}p(t)^2$$

where

$$p(t) = \frac{1 + e\cos f(t)}{1-e^2}$$

At this point, the system can be integrated numerically. Still, some interesting although limited analytic work can be done for the case $e = 0$, as first discussed below.

Zero eccentricity case

Let us first consider the simplified case in which the orbit does not have any eccentricity: $e = 0$. In that case, a closed form formula for $f(t)$ is readily found:

$$f(t) = \frac{2\pi t}{T}$$

Plugging this expression into Equation (18), and defining $\beta = \theta - f(t)$, we find

$$\ddot{\beta} = -2\frac{\epsilon^2}{T^2}\sin(2\beta)$$

Consider a situation in which the departure from synchronous rotation is small, i.e. small β: we then have

$$\ddot{\beta} + \frac{4\epsilon^2}{T^2}\beta = 0$$

This corresponds to a stable situation with small amplitude oscillations with angular frequency $2\epsilon/T$.

Finite eccentricity case

If we integrate Equation (18) numerically for the case $e = 0$ and taking the initial $\dot{theta}$ to be slightly bigger than $2\pi/T$ (say as a result of a collision with an asteroid), this is exactly the result one finds. Sussman and Wisdom considered the case of the Moon, where $\epsilon = 0.026\pi$ and took $\theta_0 = 0$, $\dot{\theta} = 2.02\pi/T$. They obtained the smooth curve shown in Figure 9. This oscillation is called the *free libration of the Moon.* We can see that the period of oscillation is about 40 lunar orbit. That agrees with the small β approximation we used above, since we had found

$$T_{free\ libration} = \frac{\pi}{\epsilon}T_{orbit} = \frac{1}{0.026}T_{orbit} \approx 38.5T_{orbit}$$

One can also integrate (18) numerically for the actual eccentricity of the orbit of the Moon around the Earth, currently around $e = 0.05$. Doing so, one finds the faster oscillating curve also shown in 9. These fast oscillations, with the period of the lunar orbit, are called the *optical libration of the Moon.*

Before closing, let us mention that there are other cases of spin -orbit resonance in the solar system and the universe, and not all of them are such that the spin period is equal to the orbit period. A famous case is the motion of Mercury around the Sun: Mercury rotates three times around its axis as it circles the Sun twice. In other words, the small oscillations we plotted for the Moon in Figure 9 would be observed for the quantity $\theta - 3/2f$. To understand why this is possible, we will have to look at phase portraits, using tools of Hamiltonian mechanics, which we will start developing in the next lecture. We will find out that the system we just studied is in fact chaotic in nature, with stability islands corresponding to spin-orbit resonances. If we drive the system hard enough, with large ϵ and e, we can even find extended regions of chaotic trajectories. A real life example of this is the chaotic motion of Hyperion, a natural satellite of Saturn.

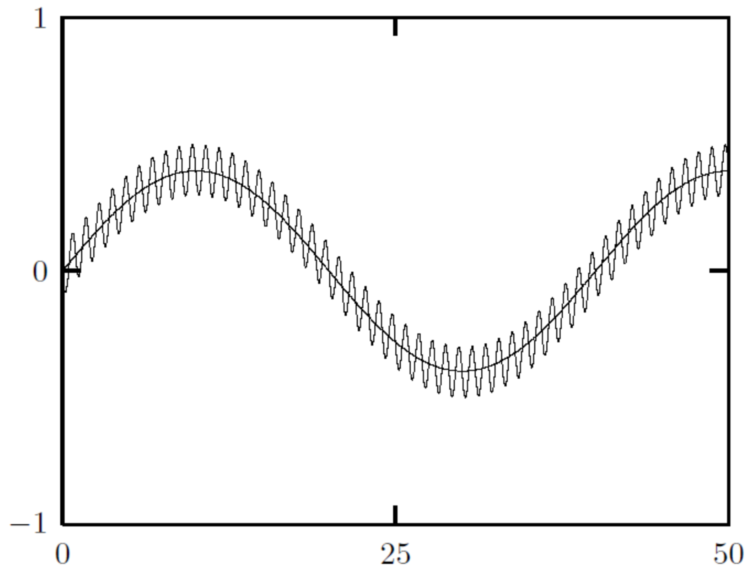

Figure 9: $\theta(t) - f(t)$ versus time for 50 orbit periods, for the case of the Moon. The inial conditions are such that $\theta = 0$ and $\dot{\theta} = 1.01\dot{f}$. The smoother curve corresponds to an orbit without eccentricity, and the more oscillatory curve corresponds to an elliptic orbit with eccentricity $e = 0.05$. Image reproduced from the textbook *Structure and Interpretation of Classical Mechanics*, MIT Press (2007), with permission of the authors.