1 Phase portraits

1.1 One dimensional system

Consider the generic one dimensional case of a point mass m described by a generalized coordinate q and subject to a time independent potential $V(q)$ such that a Lagrangian for the point mass m is

$$L(q, \dot{q}) = \frac{m}{2}\dot{q}^2 - V(q)$$

The Hamiltonian of the system is

$$H(q, p, t) = \frac{1}{2}\frac{p^2}{m} + V(q)$$

Energy is conserved, so we can write

$$\frac{1}{2}\frac{p^2}{m} = E - V(q) \qquad \Leftrightarrow \qquad p = \pm\sqrt{2m(E - V(q))}$$

Since the first canonical equation is

$$\frac{dq}{dt} = \frac{p}{m} = \pm\sqrt{\frac{2}{m}(E - V(q))}$$

the solution $q(t)$ can be expressed as a definite integral and obtained as follows

$$\int \frac{dq}{\sqrt{E - V(q)}} = \pm\left(\frac{2}{m}\right)^{1/2} \int dt$$

Even if so, the definite integral on the left can be hard to evaluate without a computer for certain $V(q)$, and may not yield as many insights as looking at the phase portrait of the system.

What exactly is a *phase portrait*? As Hamilton's equations make clear, the dynamical state of a system is completely specified by the phase state (q, p, t) given either the Lagrangian or the Hamiltonian to have the map between generalized velocities and momenta. A phase portrait is a geometric representation of the trajectories of a dynamical system in the phase space (q, p).

For the one dimensional system studied above, the phase space is two dimensional: q gives one dimension, and p the other dimension. Now, since the energy is conserved, and we could write

$$\frac{p^2}{2m} = E - V(q)$$

the orbit in phase space corresponding to a certain energy is a curve in phase space. All orbits in phase space (i.e. the phase portrait of the system) are just level curves of the Hamiltonian. In Lecture 1, we plotted the phase portrait for the simple pendulum, which we reproduce in Figure 1 as an illustration of our discussion.

The time evolution of the system in phase space is given by Hamilton's equations. Specifically, since $(\partial_1 H, \partial_2 H)$ is the gradient of the Hamiltonian, we see that Hamilton's equations tells the system to move perpendicular to the gradient of H. This is equivalent to saying that the system follows contours of constant energy, and is intuitive.

Now, there are two points in the phase portrait of the pendulum where the gradient of H vanishes: at the center of the bowl, corresponding to $(\theta, p_\theta) = (0, 0)$, and at the saddle point $(\theta, p_\theta) = (\pi, 0)$. Phase space trajectories in the neighborhood of the minimum point $(\theta, p_\theta) = (0, 0)$ remain close to that minimum point: it is a *stable equilibrium*, corresponding to the situation in which the bob is simply hanging. Phase space trajectories in the neighborhood of the saddle point $(\theta, p_\theta) = (\pi, 0)$ leave the vicinity of that equilibrium point: it is an *unstable equilibrium*, corresponding to the situation in which the bob is standing upright.

The contour that crosses the saddle point is called a *separatrix*, as it separates two regions with vastly different behavior. Inside the separatrix, the contours describe bounded oscillations about the stable equilibrium, and outside the separatrix the contours correspond to situations in which the pendulum circulates, with θ continuously increasing or decreasing without changing sign. Note finally that along a trajectory on the separatrix, the pendulum takes an infinite time to approach the saddle point. This can however not be deduced from the phase portrait.

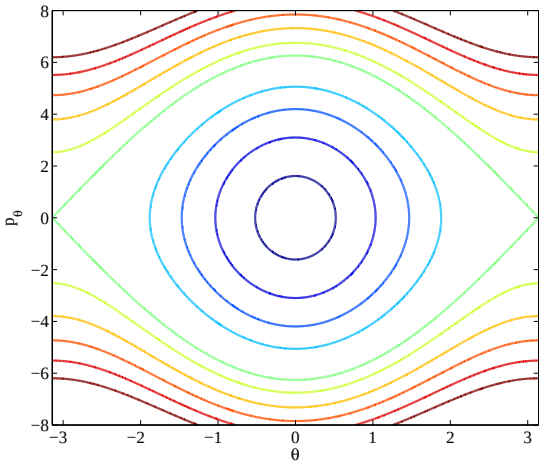

Figure 1: Phase portrait for the simple pendulum with $g = 9.8$ and $l = 1$.

1.2 Phase space reduction

Let $H(q_1, \ldots, q_N, p_1, \ldots, p_N, t)$ be a Hamiltonian for an N-dimensional problem, with a $2N$-dimensional phase space. Let us suppose that H does not depend explicitly on q_i. Hamilton's equations then tell us that p_i is conserved. Therefore, Hamilton's equations for the $2N - 2$ quantities do not depend on q_i and involves the constant p_i that only depends on initial conditions. The dimension of the problem that is left to solve has been reduced by 2. This is quite powerful. Of course, one may still want to solve for the time evolution of q_i, but once the rest of the problem is solved, all one has to do to compute $q_i(t)$ is to evaluate a definite integral.

Note that in principle, all this is also possible in the Lagrangian formulation. However, that requires replacing the $\dot{q}_i$ with their associated p_i wherever they appear, which can be cumbersome. In the Hamiltonian formulation, that happens automatically. We already saw an example of automatic phase space reduction when presenting the Hamiltonian representation of the point mass in a central potential in Section 2.3.2 of Lecture 6. Let us give another example with the axisymmetric top we studied in Lecture 4. After a bit of algebra, we had obtained the following expression for the energy, which we can call the Hamiltonian function since it only involves generalized coordinates and conjugate momenta:

$$H(\theta, p_\theta, p_\phi, p_\psi) = \frac{p_\theta^2}{2I} + \frac{(p_\phi - \cos\theta p_\psi)^2}{2I\sin^2\theta} + \frac{p_\psi^2}{2I_{33}} + mgR\cos\theta \tag{1}$$

We see that the Hamiltonian does not depend explicitly on ϕ and ψ, so p_ϕ and p_ψ are conserved quantity, as we had already proved in Lecture 4. Let us focus on the special case when $p_\phi = p_\psi = p$. By definition of the Euler angles, this equality must hold when the top is vertical. However, it is not limited to this situation, as we will now see. The Hamiltonian becomes

$$H(\theta, p_\theta) = \frac{p_\theta^2}{2I} + p^2\frac{(1-\cos\theta)^2}{2I\sin^2\theta} + \frac{p^2}{2I_{33}} + mgR\cos\theta = \frac{p_\theta^2}{2I} + V_{eff}(\theta) \tag{2}$$

with

$$V_{eff}(\theta) = p^2\frac{(1-\cos\theta)^2}{2I\sin^2\theta} + \frac{p^2}{2I_{33}} + mgR\cos\theta$$

As one can see in Figure 2, for p large, V_{eff} has a unique minimum at $\theta = 0$. For p small, there is a maximum at $\theta = 0$, and two symmetric minima on each side of $\theta = 0$. The transition occurs for $p = p_c$ such that the minimum of V_{eff} at $\theta = 0$ becomes a maximum. Computing the second derivative of V_{eff}, we easily find the condition

$$p_c^2 = 4ImgR \qquad \Rightarrow p_c = \sqrt{4ImgR}$$

At $\theta = 0$, we have $p = I_{33}\omega$, where ω is the rotation rate of the spinning motion of the top. p_c corresponds to the following critical rotation:

$$\omega_c = \sqrt{4IgmR}/I_{33}$$

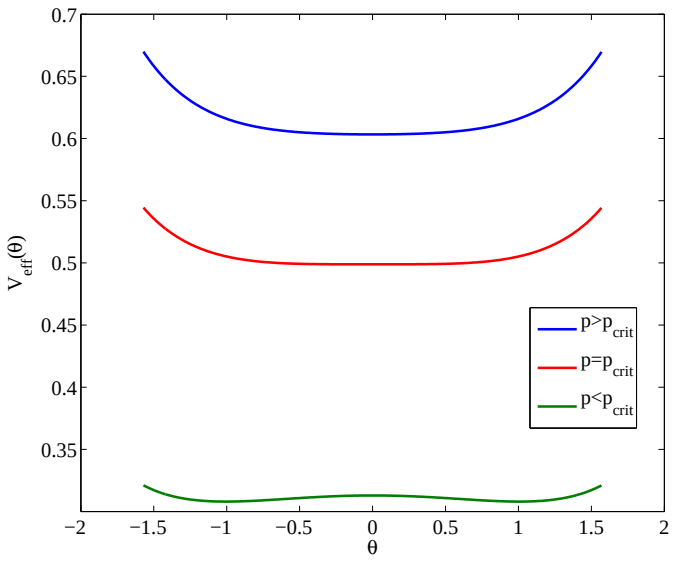

Figure 2: $V_{eff}(\theta)$ as a function of θ for different values of p

For $\omega > \omega_c$, the top is stable standing vertically, as can be seen from the phase portrait shown in Figure 3, obtained for $\omega > \omega_c$. For $\omega < \omega_c$, the top is unstable standing vertically, as can be seen from the phase portrait shown in Figure 4, obtained for $\omega < \omega_c$.

2 Phase space evolution and Liouville's theorem

Most problems do not have enough symmetries and associated conserved quantities to be reducible to quadratures or to situations in which the phase portraits are just given by contour lines corresponding the value of a conserved quantity such as the energy. In general, one thus has to solve the problem numerically by integrating the canonical equations in time. Even if so, phase portraits can be constructed numerically that yield plenty of insight on the dynamics of the system, as we will soon see when we discuss Poincaré plots. Before we do so, we present some important properties of the evolution of a Hamiltonian system in phase space, and introduce the simple mechanical system we will use to illustrate these properties, namely the periodically driven pendulum.

2.1 The periodically driven pendulum

Consider a pendulum of length l and mass m supported by a pivot that is driven in the vertical direction by a given function of time $y_D(t)$, as shown in Figure 5. The position of the bob in the Cartesian coordinate system $x - y$ shown in the figure is

$$x = l \sin\theta \qquad y = y_D(t) - l\cos\theta$$

The velocity of the bob therefore is

$$v_x = l\dot{\theta}\cos\theta \qquad v_y = \frac{dy_D}{dt} + l\dot{\theta}\sin\theta$$

We thus see that the kinetic energy T of the bob can be expressed in terms of θ, $\dot{\theta}$ and t as

$$T(\theta,\dot{\theta},t) = \frac{m}{2}\left[l^2\dot{\theta}^2 + \left(\frac{dy_D}{dt}\right)^2 + 2l\dot{\theta}\frac{dy_D}{dt}\sin\theta\right]$$

The potential energy can also be written in terms of θ, $\dot{\theta}$ and t:

$$V(\theta,\dot{\theta},t) = mg\left(y_D(t) - l\cos\theta\right)$$

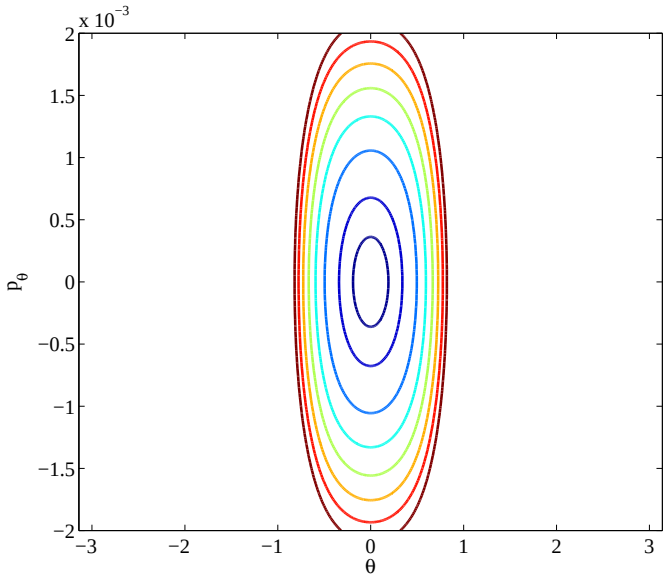

Figure 3: Phase portrait (θ, p_θ) for the axisymmetric top with $p_\phi = p_\psi$, $I = 3.28{\times}10^{-5}$ kg m^2, $I_{33} = 6.6{\times}10^{-5}$ kg m^2, $mgR = 4.56 \times 10^{-2}$ kg m^{2}s^{-2} and $\omega = 130$ rad/s$> \omega_c \approx$ rad/s

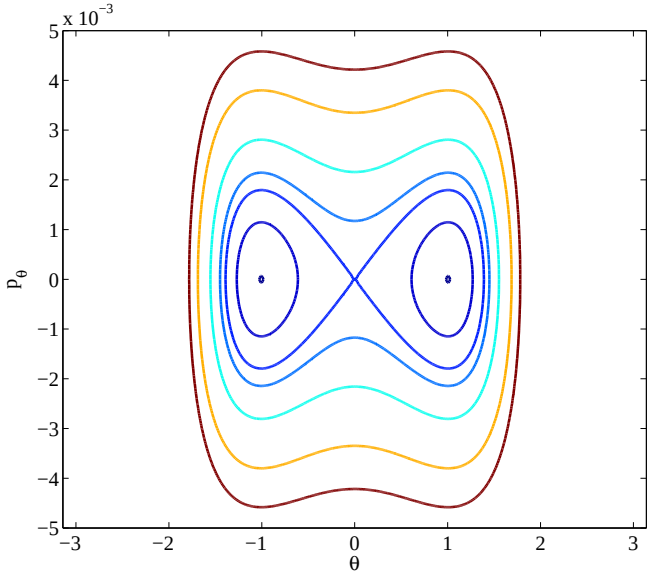

Figure 4: Phase portrait (θ, p_θ) for the axisymmetric top with $p_\phi = p_\psi$, $I = 3.28{\times}10^{-5}$ kg m^2, $I_{33} = 6.6{\times}10^{-5}$ kg m^2, $mgR = 4.56 \times 10^{-2}$ kg m^{2}s^{-2} and $\omega = 90$ rad/s$< \omega_c \approx$ rad/s

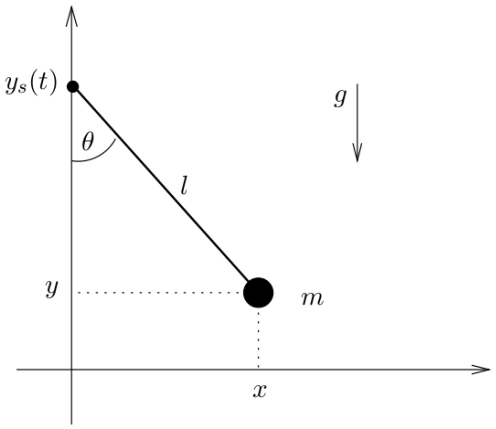

Figure 5: Periodically driven pendulum. Image reproduced from the textbook *Structure and Interpretation of Classical Mechanics*, MIT Press (2007), with permission of the authors. This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License. To view a copy of this license, visit creativecommons.org.

where the reference axis $V = 0$ was taken to coincide with the x-axis. By Hamilton's principle, a Lagrangian for the driven pendulum is

$$L(\theta, \dot{\theta}, t) = T - V = \frac{m}{2}\left[l^2\dot{\theta}^2 + \left(\frac{dy_D}{dt}\right)^2 + 2l\dot{\theta}\frac{dy_D}{dt}\sin\theta\right] - mg\left(y_D(t) - l\cos\theta\right)$$

We compute the Legendre transformation of L with respect to the $\dot{\theta}$ variable to obtain a Hamiltonian for the driven pendulum. After a straightforward calculation, we find

$$\begin{aligned} H(\theta, p_\theta, t) = &-\frac{m}{2}\left[\frac{1}{m^2l^2}\left(p_\theta - ml\sin\theta\frac{dy_D}{dt}\right)^2 + \frac{2}{ml}\left(p_\theta - ml\sin\theta\frac{dy_D}{dt}\right)\frac{dy_D}{dt}\sin\theta + \left(\frac{dy_D}{dt}\right)^2\right] \\ &+ \frac{p_\theta}{ml^2}\left(p_\theta - ml\sin\theta\frac{dy_D}{dt}\right) + mg(y_D - l\cos\theta) \end{aligned}$$

In this course, we will always consider the case of the periodically driven pendulum, where $y_D = A\cos(\omega t)$ with A the drive amplitude and ω the drive frequency. In that case, the Hamiltonian simplifies somewhat, to the following expression

$$H(\theta, p_\theta, t) = \frac{p_\theta^2}{2ml^2} + \frac{A\omega}{l}\sin(\omega t)\sin\theta p_\theta - \frac{mA^2\omega^2}{2}\sin^2(\omega t)\cos^2\theta + mgA\cos(\omega t) - mgl\cos\theta$$

This is the Hamiltonian function we will use in the canonical equations, which we will solve numerically.

2.2 Phase space evolution

The first idea one might have to try and understand the key features of a given Hamiltonian system that is not integrable is to solve the canonical equations numerically, and to plot in phase space the values of the generalized coordinates and conjugate momenta obtained at each time step. For N-dimensional systems with $N \geq 2$, it is not possible to represent the phase portait in one plot. People thus usually plot the separate phase portraits (q_1, p_1), (q_2, p_2),... for each of the N dimensions of the problem separately. For the case of the periodically driven pendulum, we do not run into this issue, and will always be able to represent all we want to represent in the two-dimensional (θ, p_θ) space.

In Figure 6, we show the phase space trajectory for the periodically driven pendulum with mass $m = 1$ kg, length $l = 1$ m, $g = 9.8$ m.s^{-2} and initial conditions $(\theta_0, p_{\theta 0}) = (1, 0)$, $A = 0.1$ and $\omega = 2\sqrt{g/l}$. We recognize some interesting features, reminiscent of the phase space structure of the simple, undriven pendulum.

While interesting, Figure 6 only describes the behavior of a single trajectory, with fairly arbitrary initial conditions. It fails to give an idea of all the possible dynamics of the system if the conditions were to be

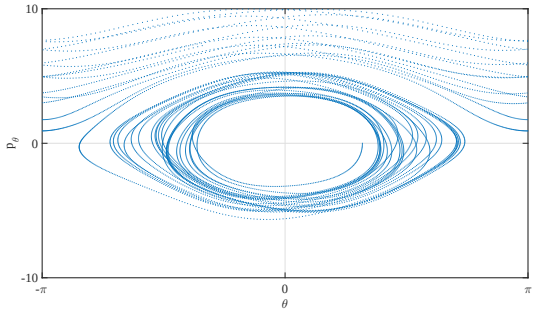

Figure 6: Phase space trajectory for the trajectory with initial conditions $(\theta_0, p_{\theta 0}) = (1, 0)$ of a driven pendulum with mass $m = 1$ kg, length $l = 1$ m, $g = 9.8$ m.s^{-2}, $A = 0.1$ and $\omega = 2\sqrt{g/l}$

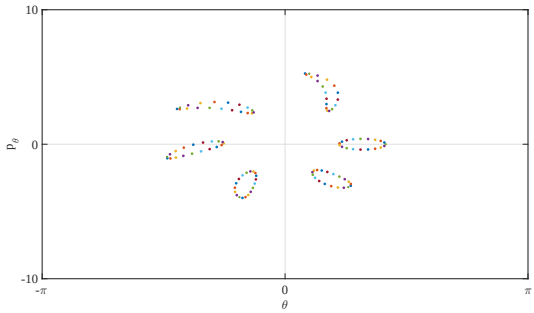

Figure 7: Phase space trajectories at 6 different equispaced times for 20 trajectories of a driven pendulum with initial conditions as given by the ellipse on the $p_\theta = 0$ axis on the right hand side of the figure. The driven pendulum here has mass $m = 1$ kg, length $l = 1$ m, $g = 9.8$ m.s^{-2}, $A = 0.1$ and $\omega = 2\sqrt{g/l}$.

varied. A fairly intuitive idea, in that context, is to consider an *ensemble* of initial conditions and look at the evolution of the corresponding ensemble of trajectories. This is what we show in Figure 7, in which we have plotted the position in phase space at 6 different times, including $t = 0$, of 20 trajectories corresponding to 20 different initial conditions (the initial conditions correspond to the original ellipse on the $p_\theta = 0$ axis on the right hand side of the figure). We can see that the region contained inside the 20 dots is changes shape as it is being advected and sheared by the flow in phase space. It turns out that the flow *is such that the area of the region is conserved for all times*. This is called *Liouville's theorem*, which we now prove.

2.3 Liouville's Theorem

Consider a generalized coordinate q and its conjugate momentum p. We start by deriving the general expression for the change of the infinitesimal phase space area $dqdp$ as it evolves following the flow ($\frac{dq}{dt} = v_q(q, p, t)$, $\frac{dp}{dt} = v_p(q, p, t)$). We consider an initial infinitesimal region in phase space with area $dA = dqdp = \overrightarrow{AB} \times \overrightarrow{AC}$ as shown in Figure 8. Let us consider a time dt later. The region has moved and changed shape following the phase space flow. Its area is given by $dA' = \overrightarrow{A'B'} \times \overrightarrow{A'C'}$. To lowest order in dt, the coordinates of the point A' are obtained from the coordinates of A as follows:

$$A' = A + v_q(q, p, t)dt\mathbf{e}_q + v_p(q, p, t)dt\mathbf{e}_p$$

where $\mathbf{e}_q$ and $\mathbf{e}_p$ are the unit vectors in the q and p direction respectively. Likewise, we have

$$B' = B + v_q(q+dq, p, t)\mathbf{e}_q + v_p(q+dq, p, t)dt\mathbf{e}_p = B + v_q(q, p, t)dt\mathbf{e}_q + \frac{\partial v_q}{\partial q}dqdt\mathbf{e}_q + v_p(q, p, t)dt\mathbf{e}_p + \frac{\partial v_p}{\partial q}dqdt\mathbf{e}_p + O(dq^2, dp^2)$$

and

$$C' = C + v_q(q, p+dp, t)\mathbf{e_q} + v_p(q, p+dp, t)dt\mathbf{e}_p = C + v_q(q, p, t)dt\mathbf{e}_q + \frac{\partial v_q}{\partial p}dpdt\mathbf{e}_q + v_p(q, p, t)dt\mathbf{e}_p + \frac{\partial v_p}{\partial p}dpdt\mathbf{e}_p + O(dq^2, dp^2)$$

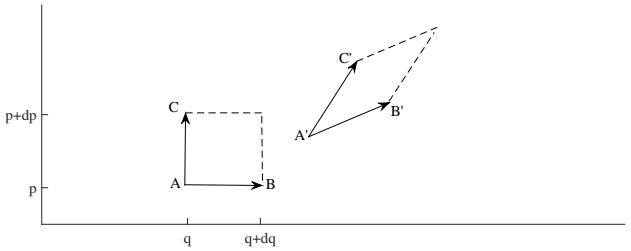

Figure 8: Change of an infinitesimal volume advected in a given two-dimensional flow

Hence,

$$\overrightarrow{A'B'} \times \overrightarrow{A'C'} = \overrightarrow{AB} \times \overrightarrow{AC} + \left(\frac{\partial v_q}{\partial q} + \frac{\partial v_p}{\partial p} \right) dqdpdt + O(dt^2, dp, dq)$$

In other words,

$$dA' = dA + \nabla \cdot \mathbf{v} dqdpdt + O(dt^2)$$

where $\mathbf{v}$ is the phase space flow. We conclude that the time change of an infinitesimal phase space region $dqdp$ is given by the formula

$$\frac{d}{dt}(dqdp) = \nabla \cdot \mathbf{v} dqdp \tag{3}$$

When the dynamics of the system is given by a Hamiltonian,

$$v_q = \dot{q} = \frac{\partial H}{\partial p} \qquad v_p = \dot{p} = -\frac{\partial H}{\partial q}$$

so

$$\nabla \cdot \mathbf{v} = \frac{\partial v_q}{\partial q} + \frac{\partial v_p}{\partial p} = \frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q} = 0 \tag{4}$$

We see that the *canonical phase space flow is always non divergent*. Going back to Eq. (3), this implies that the area of an infinitesimal phase space region is always conserved by the flow, and integrating over larger regions, that the area of any closed region in phase space is conserved by the canonical phase space flow. This is Liouville's theorem.

Liouville's theorem is not limited to 1-dimensional problems. It also applies to N-dimensional Hamiltonian systems. The phase space volume element then is $dV = dq_1 dp_1 \dots dq_N dp_N$, and the same proof applies since the phase-flow divergence is zero for every conjugate pair (q_i, p_i).

Note lastly that we have not assumed any particular form for the Hamiltonian. In particular, Liouville's theorem applies to systems for which the Hamiltonian depends explicitly on the time variable and energy is not conserved. We chose the example of the driven pendulum for that purpose.

2.4 Poincaré recurrence theorem

An important corollary of Liouville's theorem is the *Poincaré recurrence theorem*. It applies to any Hamiltonian system for which the phase space is a bounded domain D. Its statement is as follows:

Consider an initial point $\mathbf{x}_0$ in bounded phase space. For any open finite neighborhood U of $\mathbf{x}_0$ that we choose, there are trajectories which emanate from initial points in that neighborhood that eventually return to this neighborhood.

The proof is quite intuitive. Consider discrete time steps Δt, and M the map that takes U to its new region $M(U)$ in phase space, following the Hamiltonian trajectories for an amount of time Δt. Consider the countable set $\left\{M^i(U)\right\}_{i=1,2,\dots}$ of the succesive maps of U. Since D has a finite volume, there exists a j and a k, with $j < k$ such that $M^j(U)$ intersects $M^k(U)$. Indeed, one cannot fit infinitely many nonintersecting finite volumes into the bounded domain D. Now, if $M^j(U)$ intersects $M^k(U)$, then $M^{j-1}(U)$ intersects $M^{k-1}(U)$ since $M^{-1}(M^j(U) \cap M^k(U)) = M^{j-1}(U) \cap M^{k-1}(U)$. Following the same argument, $M^{j-2}(U)$ intersects $M^{k-2}(U)$. Pushing this logic to its end, $M^{k-j}(U)$ intersects U. QED.

The proof highlights the following very potent result: for any point $\mathbf{x_0}$ in a bounded phase space and any neighborhood U of this point, there exists a subneighborhood such that the trajectories emanating from all of the points in that subneighborhood return to that subneighborhood. In other words, almost every trajectory returns arbitrarily close to where it started. This is a strong and disturbing physical result. Let us spend a few moments explaining why it may be physically disturbing, for those of you that are not ready to give up mathematical beauty in favor of physical hand-waviness.

First, it is worthwhile to stress that the condition of boundedness of the domain D in the theorem is not a condition that makes the theorem only applicable to a few very artificial physical systems. It is quite the opposite. Indeed, consider point masses such as particles in a closed container. Clearly, the configuration space coordinates describing their positions is bounded. If in addition energy is conserved, the conjugate momenta are also bounded, and so is phase space.

Now, let us see why Poincaré's recurrence theorem may hurt our physical intuition and contradict other theorems of physics we may know. Consider a set of N gas particles inside a container. The dimension of phase space for this system is $6N$. Imagine that at time t_0, they all occupy a very small region of the container, and have arbitrary, finite velocities. This set of initial conditions corresponds to a point $\mathbf{x}_0$ in the 6N-dimensional phase space. Poincaré's recurrence theorem then says that there are initial conditions in the neighborhood of $\mathbf{x}_0$ such that trajectories corresponding to these initial conditions return to the neighborhood in a finite amount of time. That means that for these initial conditions, the particles initially in the small region of the container all return to that region in finite time, and return again, and again. This fact, appearing to contradict the second principle of thermodynamics (but only appearing to) has bothered physicists and philosophers for a long time.

2.5 Digression: a brief discussion of kinetic theory

Some physical systems, such as a very low density gas or a very hot, low density plasma of positively charged ions and electrons have too low a collision frequency and too high a mean free path for the fluid approximation to hold. The Navier-Stokes equations (or equivalent Magnetohydrodynamic equations for plasmas) do not describe the system accurately. In these conditions, an intuitive idea is to solve the coupled set of N second order differential equations given by Newton's laws, or $2N$ first order canonical equations to determine the evolution of the N particles in the system. For N not too large, $N \leq 10^9$ this may be numerically feasible nowadays. However, for the typical number of particles studied in a magnetic fusion plasma or in some rarefied gases, $N > 10^{20}$ is common and untractable numerically.

For these situations, a statistical treatment of the dynamics of the system in phase space is required. This statistical treatment is often known as *kinetic theory*, and can be seen as an intermediate between a particle-by-particle treatment and a fluid theory.

In kinetic theory, the evolution of the system is described in terms of a continuous density function $f(\mathbf{q}, \mathbf{p}, t)$ such that $f(\mathbf{q}, \mathbf{p}, t) d\mathbf{q} d\mathbf{p}$ is the number of particles in the infinitesimal volume $d\mathbf{q}d\mathbf{p}$. Once $f(\mathbf{q}, \mathbf{p}, t)$ is computed (solving an equation we have not yet stated), macroscopic quantities about the system are obtained by taking moments of the distribution function. For example, the particle density n is obtained from the zeroth moment:

$$n(\mathbf{q}, t) = \iiint_{\mathbf{p}_{min}}^{\mathbf{p}_{max}} f(t, \mathbf{q}, \mathbf{p}, t) d\mathbf{p}$$

The mean momentum $\mathbf{P}$ (which is often directly related to the mean fluid velocity $\mathbf{V}$) is given by the first moment

$$n\mathbf{P} = \iiint_{\mathbf{p}_{min}}^{\mathbf{p}_{max}} \mathbf{p} f(t, \mathbf{q}, \mathbf{p}, t) d\mathbf{p}$$

Higher moments would give the temperature of the system, etc.

What is the equation determining the evolution of f? Consider N_p particles inside an infinitesimal volume $d\mathbf{q}d\mathbf{p}$ in phase space. In the absence of chemical or nuclear reactions, or any reaction that may cause the creation or destruction of particles, which we assume are absent from the system under study, the number of particles N_p as one follows the particles' trajectory:

$$\frac{dN_p}{dt} = 0 \qquad \Leftrightarrow \qquad \frac{d}{dt}(f(\mathbf{q}, \mathbf{p}, t) d\mathbf{q} d\mathbf{p}) = 0$$

We know from Liouville's theorem that

$$\frac{d}{dt}\left(d\mathbf{q}d\mathbf{p}\right) = 0$$

Therefore, the equation for the evolution of f is

$$\frac{df}{dt} = 0$$

Using the Hamiltonian formulation introduced in section **??**, this takes the following simple and explicit form:

$$\partial_3 f + \{f, H\} = 0 \tag{5}$$

Illustration: Kinetic equation for a hot plasma in the absence of magnetic fields Consider charged particles with charge q subject to self-electric fields. The single particle Hamiltonian for such a system is

$$H(\mathbf{x}, \mathbf{p}, t) = \frac{\mathbf{p}^2}{2m} + q\phi$$

We have

$$\nabla_{\mathbf{p}} H = \frac{\mathbf{p}}{m} \qquad \nabla_x H = q\nabla_{\mathbf{x}}\phi$$

so Equation (5) becomes the following evolution equation for $f(\mathbf{q}, \mathbf{p}, t)$:

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{x}} f - q\nabla_{\mathbf{x}}\phi \cdot \nabla_{\mathbf{p}} f = 0 \tag{6}$$

Eq. 6, along with Poisson's equation for the self-consistent electric potential

$$\nabla^2_{\mathbf{x}}\phi = -\frac{q}{\epsilon_0} \iiint_{\mathbf{p}_{min}}^{\mathbf{p}_{max}} f(t, \mathbf{x}, \mathbf{p}, t) d\mathbf{p} \tag{7}$$

form what is often called the *Vlasov-Poisson system.*