The Frequency Domain

DS-GA 1013 / MATH-GA 2824 Optimization-based Data Analysis

http://www.cims.nyu.edu/~cfgranda/pages/OBDA_fall17/index.html

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Fourier Representations

Sampling theorem

Convolution

Wiener deconvolution

Complex Exponential

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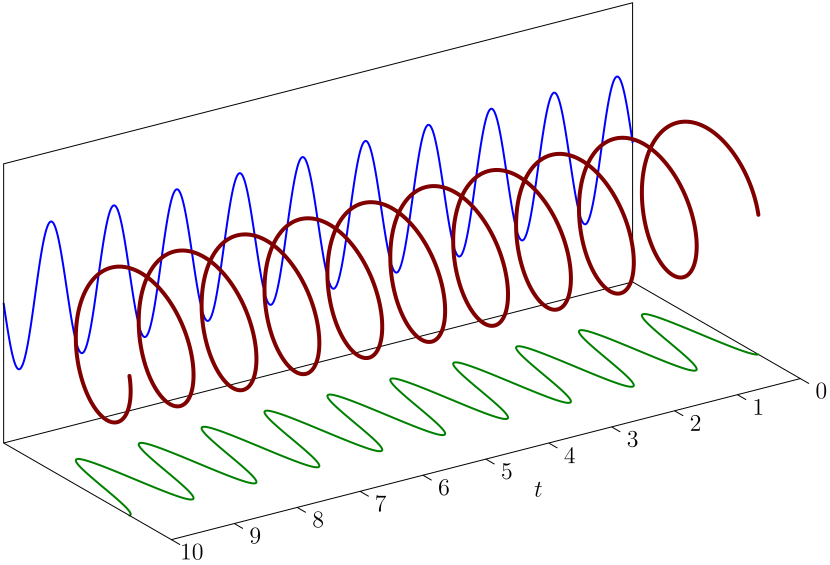
Complex Exponential: $\exp(2\pi ix)$ for $x \in [0, 10]$



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4. The family of functions h_k, for $k \in \mathbb{Z}$, form an orthonormal set of functions in $\mathcal{L}_2[-1/2, 1/2]$

Proof: h_k have unit norm

Note that

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&= \int_{-1/2}^{1/2} \exp(i 2\pi (j-k) t) \, \mathrm{d}t \\
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&= \left[\frac{\exp(i2\pi(j-k)t)}{i2\pi(j-k)}\right]_{-1/2}^{1/2} \\
&= \frac{\exp(i\pi(j-k)) - \exp(-i\pi(j-k))}{i2\pi(j-k)}
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&= 0.
\end{aligned}$$

Fourier Series

1. We assume $f : [-1/2, 1/2] \to \mathbb{C}$ with $f \in \mathcal{L}_2[-1/2, 1/2]$ and $f(-1/2) = f(1/2)$. This corresponds to 1-periodic functions.
2. We define the Fourier series of f, denoted $S\{f\}$, by

$$S\{f\} := \sum_{k \in \mathbb{Z}} F[k] h_k,$$

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4. Convergence of Fourier Series:

$$\left\| f - \sum_{k=-n}^{n} F[k] h_k \right\|_{\mathcal{L}_2} \to 0,$$

   as $n \to \infty$. If f is also continuously differentiable the convergence is uniform.

Properties of Fourier Series

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F[k]h_k(t) + F[-k]h_{-k}(t) &= F[k]h_k(t) + \overline{F[k]h_k(t)} \\
&= 2\,\mathrm{Re}\left(F[k]h_k(t)\right)
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$$\begin{aligned}
F[k]h_k(t) + F[-k]h_{-k}(t) &= F[k]h_k(t) + \overline{F[k]h_k(t)} \\
&= 2\,\mathrm{Re}\left(F[k]h_k(t)\right) \\
&= 2\,\mathrm{Re}\left(F[k](\cos(2\pi kt) + i\sin(2\pi kt))\right)
\end{aligned}$$

Properties of Fourier Series

If $f : [-1/2, 1/2] \to \mathbb{R}$ then

$$\begin{aligned} S\left\{f\right\}(t) &= \sum_{k=0}^{\infty} a_k \cos(2\pi kt) + b_k \sin(2\pi kt), \\ a_0 &:= \langle f, 1\rangle, \\ a_k &:= 2\langle f, \cos(2\pi kt)\rangle, \quad k > 0 \\ b_k &:= 2\langle f, \sin(2\pi kt)\rangle, \quad k \geq 0. \end{aligned}$$

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$$2\,\mathrm{Re}\,(F[k])\cos(2\pi kt) - 2\,\mathrm{Im}\,(F[k])\sin(2\pi kt) = a_k \cos(2\pi kt) + b_k \sin(2\pi kt).$$

Examples of Fourier Series: Gaussian

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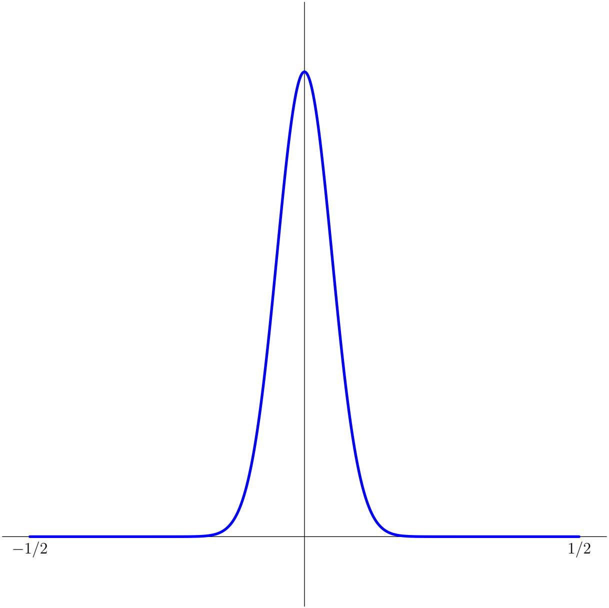
Examples of Fourier Series: Gaussian Plot $g(t)$



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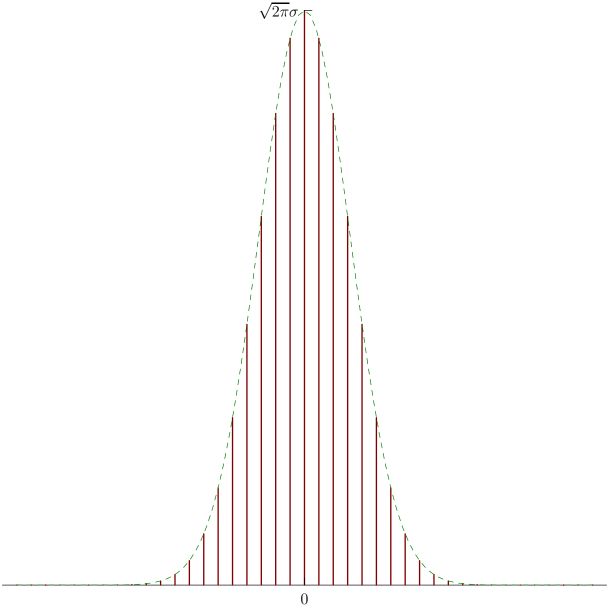

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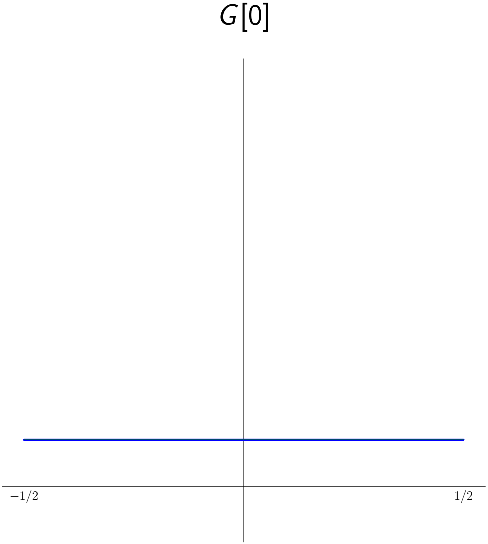

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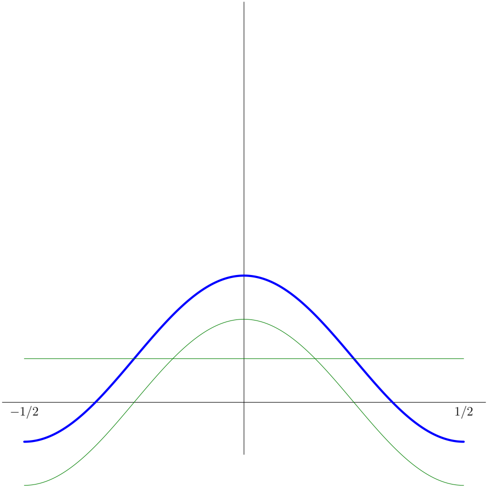
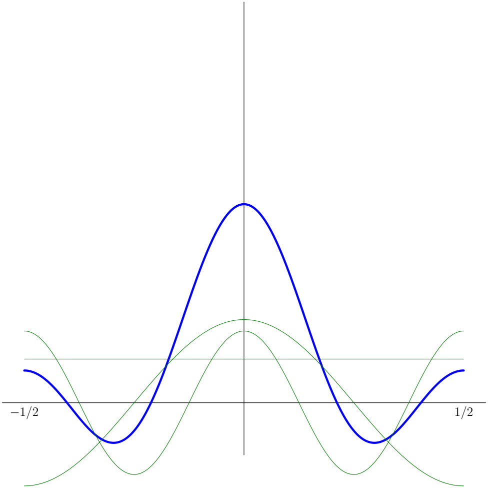
$$G[0] + G[1]e^{2\pi it} + G[-1]e^{-2\pi it} = G[0] + 2G[1]\cos(2\pi t)$$



Examples of Fourier Series: Gaussian Plot $g(t)$

$$\sum_{k=-2}^{2} G[k]e^{2\pi ikt} = G[0] + \sum_{k=1}^{2} 2G[k]\cos(2\pi kt)$$



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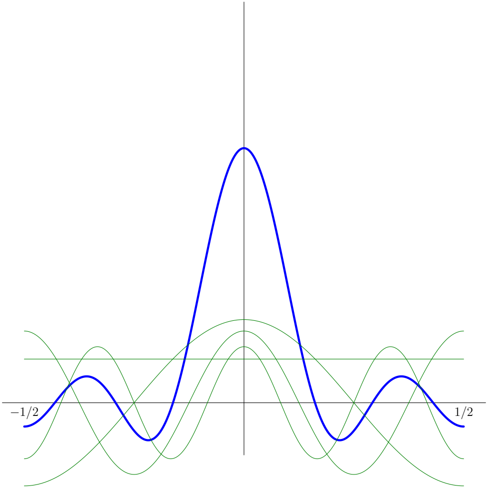

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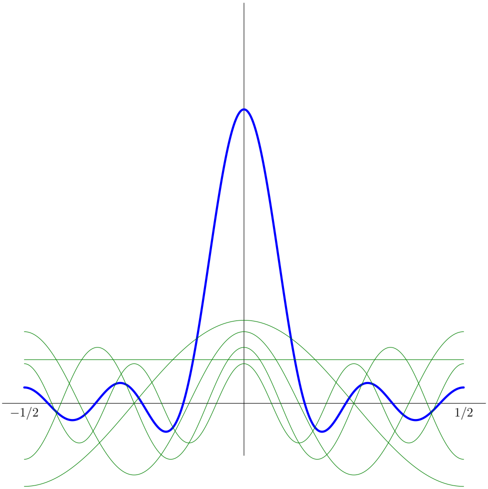
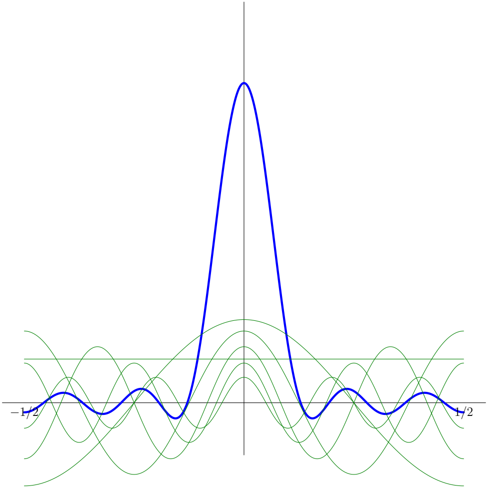


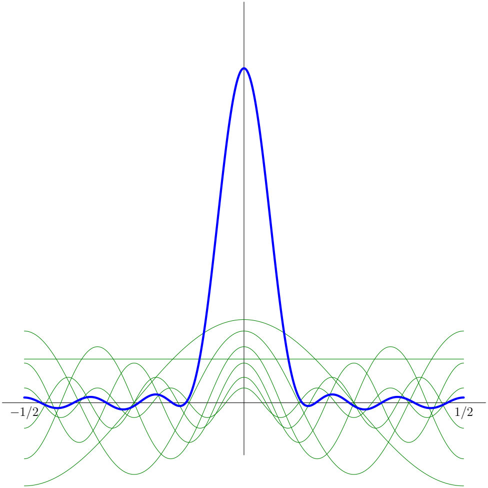
Examples of Fourier Series: Gaussian Plot $g(t)$

$$\sum_{k=-5}^{5} G[k]e^{2\pi ikt} = G[0] + \sum_{k=1}^{5} 2G[k]\cos(2\pi kt)$$



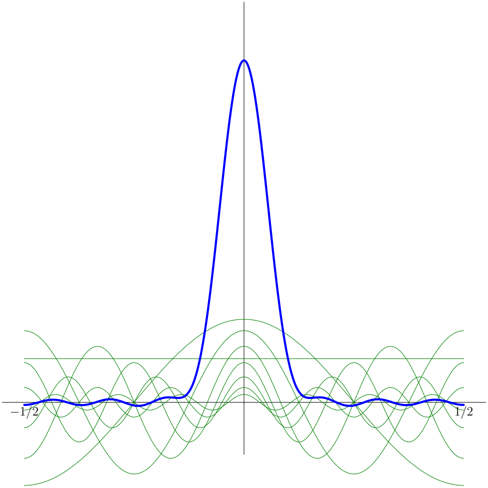
Examples of Fourier Series: Gaussian Plot $g(t)$

$$\sum_{k=-6}^{6} G[k]e^{2\pi ikt} = G[0] + \sum_{k=1}^{6} 2G[k]\cos(2\pi kt)$$



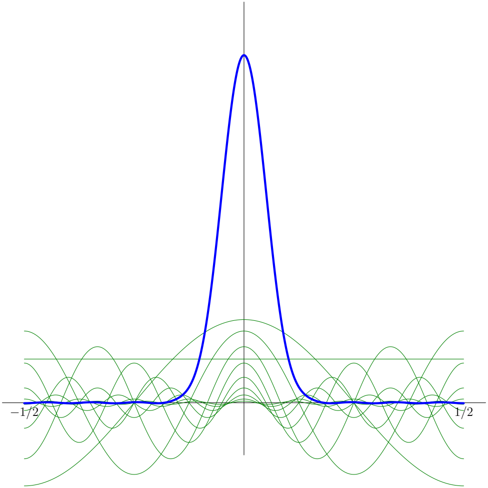
Examples of Fourier Series: Gaussian Plot $g(t)$

$$\sum_{k=-7}^{7} G[k]e^{2\pi ikt} = G[0] + \sum_{k=1}^{7} 2G[k]\cos(2\pi kt)$$



Examples of Fourier Series: Gaussian Plot $g(t)$

$$\sum_{k=-9}^{9} G[k]e^{2\pi ikt} = G[0] + \sum_{k=1}^{9} 2G[k]\cos(2\pi kt)$$



Examples of Fourier Series: Gaussian Plot $g(t)$

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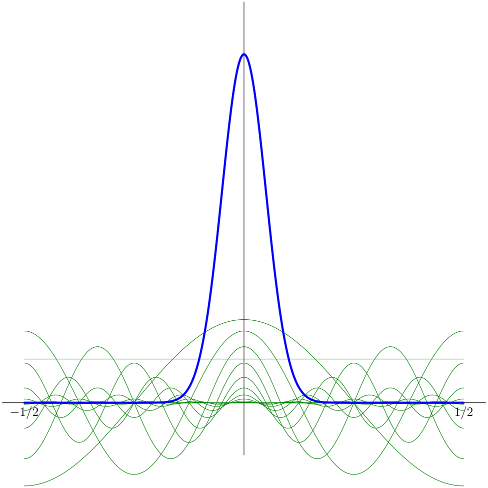

Examples of Fourier Series: Dirichlet Kernel

1. Consider a function $d_{k_c}(t)$ with Fourier coefficients $D_{k_c}(k)$ given by

$$D_{k_c}[k] = \begin{cases} 1 & \text{if } |k| \leq k_c, \\ 0 & \text{otherwise.} \end{cases}$$

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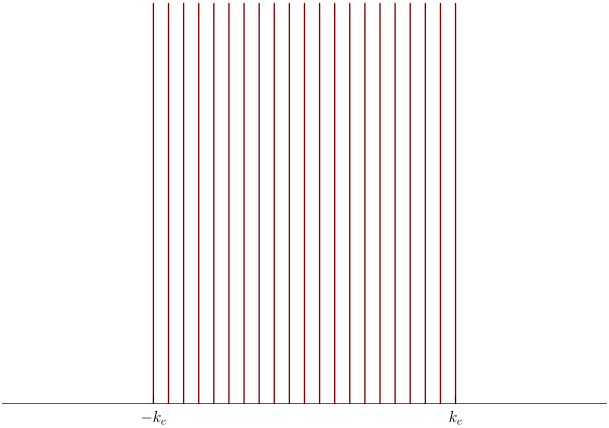
3. We have the expression

$$d_{k_c}(t) = \frac{\sin((2k_c + 1)\pi t)}{\sin(\pi t)},$$

   for $t \neq 0$ and $d_{k_c}(0) = 2k_c + 1$.

Examples of Fourier Series: Dirichlet Plot $D_{k_c}[k]$



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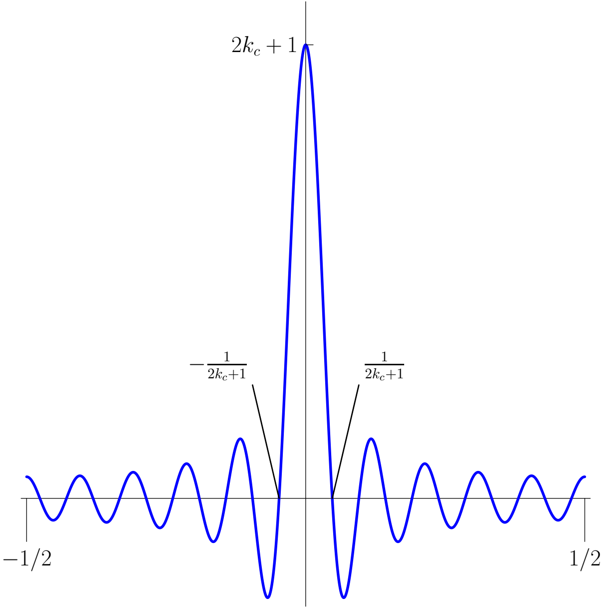

Discrete Complex Sinusoids

1. Define the discrete complex sinusoid $\vec{h}_k^{[n]} \in \mathbb{C}^n$ by

$$\vec{h}_k^{[n]}[j] := \exp\left(\frac{i2\pi kj}{n}\right),$$

for $j = 0, \dots, n-1$.

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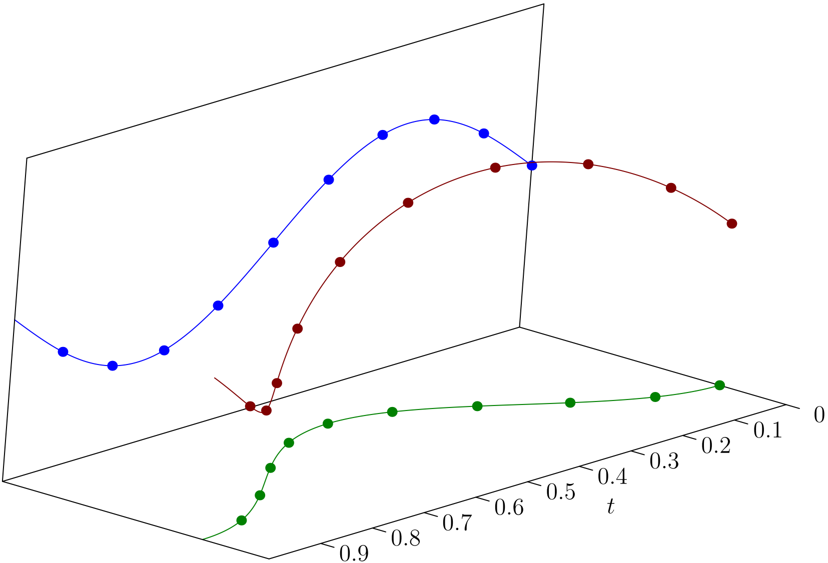
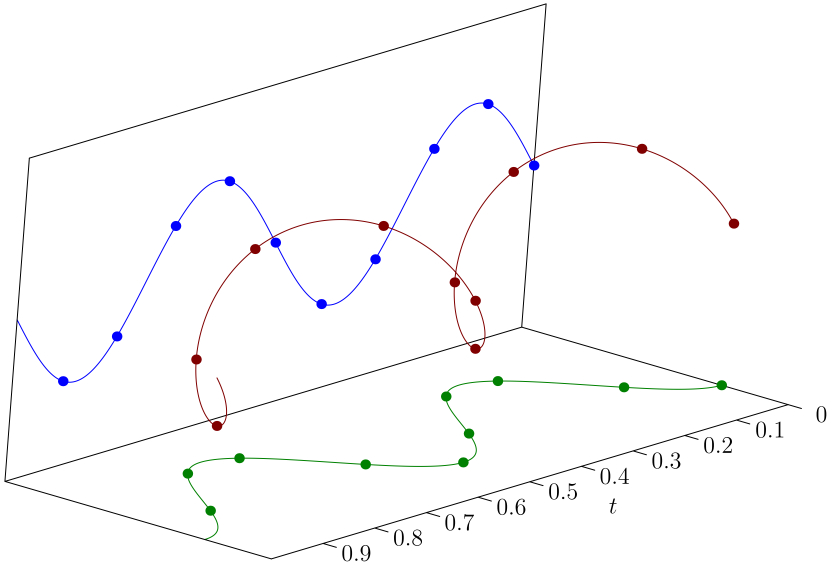
3. Formed by sampling h_k with every $1/n$ time units.

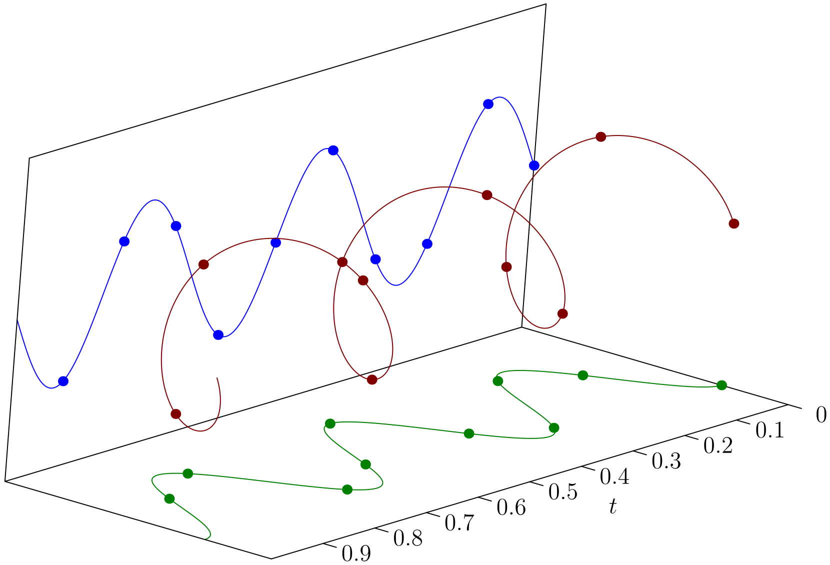
Discrete Complex Sinusoids: $\vec{h}_1^{[10]}$



Discrete Complex Sinusoids: $\vec{h}_2^{[10]}$



Discrete Complex Sinusoids: $\vec{h}_3^{[10]}$



Discrete Complex Sinusoids: Orthogonal

The discrete complex sinusoids $\frac{1}{\sqrt{n}}\vec{h}_0^{[n]}, \ldots, \frac{1}{\sqrt{n}}\vec{h}_{n-1}^{[n]}$ for an orthonormal basis of $\mathbb{C}^n$.

Discrete Complex Sinusoids: Unit norm

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$$\left\|\vec{h}_k^{[n]}\right\|_2^2$$

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$$= \sum_{j=0}^{n-1} 1$$

Discrete Complex Sinusoids: Unit norm

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$$\left\|\vec{h}_k^{[n]}\right\|_2^2 = \sum_{j=0}^{n-1} \left|\vec{h}_k^{[n]}[j]\right|^2$$

$$= \sum_{j=0}^{n-1} 1$$

$$= n.$$

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Proof. If $k \neq l$,

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$$= \sum_{j=0}^{n-1} \exp\left(\frac{i2\pi(k-l)j}{n} \right)$$

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$$
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&= \sum_{j=0}^{n-1} \exp\left(\frac{i2\pi(k-l)j}{n}\right) \\
&= \frac{1-\exp\left(\frac{i2\pi(k-l)n}{n}\right)}{1-\exp\left(\frac{i2\pi(k-l)}{n}\right)} \qquad \text{(geometric sum)}
\end{aligned}
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&= 0
\end{aligned}$$

Properties of Discrete Complex Sinusoids: DFT

1. Any vector of samples $\vec{x} \in \mathbb{C}^n$ can be written as a linear combination of the orthonormal basis vectors $\frac{1}{\sqrt{n}}\vec{h}_0^{[n]}, \ldots, \frac{1}{\sqrt{n}}\vec{h}_{n-1}^{[n]}$:

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3. Define the DFT (Discrete Fourier Transform) matrix $W \in \mathbb{C}^{n \times n}$ by

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4. Note that $\frac{1}{\sqrt{n}}W$ is unitary.
5. $\vec{x} = \frac{1}{n}W^*\vec{X}$ and $\vec{X} = W\vec{x}$

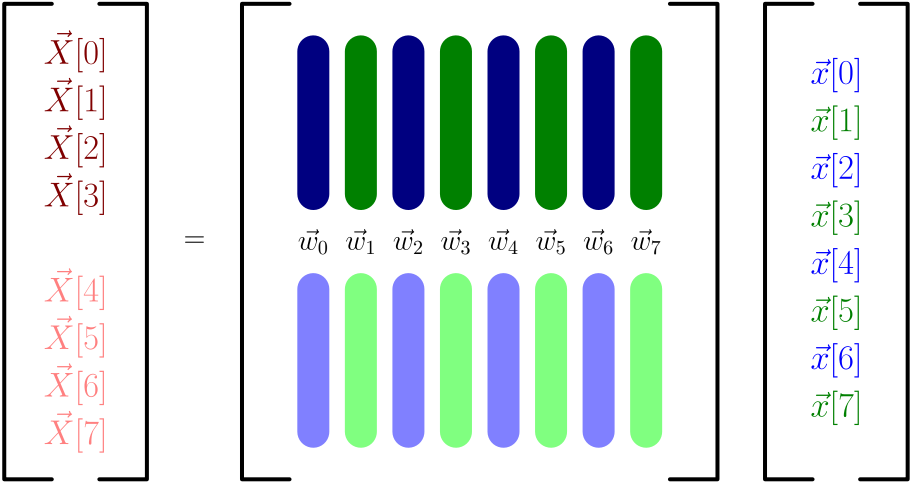
Properties of Discrete Complex Sinusoids: DFT

1. Runtime to apply DFT matrix to $\vec{x}$ to obtain $\vec{X}$ is $O(n^2)$.

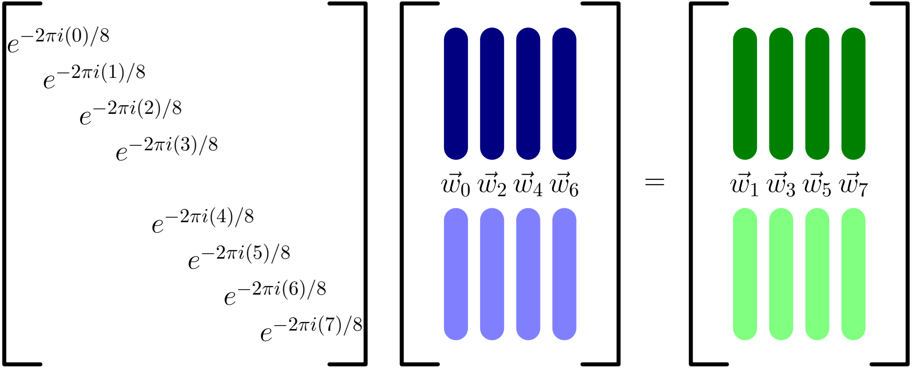
Properties of Discrete Complex Sinusoids: DFT

1. Runtime to apply DFT matrix to $\vec{x}$ to obtain $\vec{X}$ is $O(n^2)$.
2. How can we improve this time? Exploit symmetry.

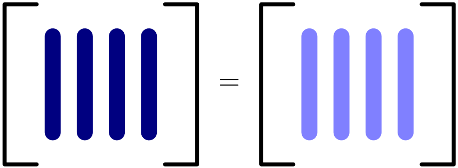
Discrete Complex Sinusoids: FFT



Discrete Complex Sinusoids: FFT



Discrete Complex Sinusoids: FFT



Discrete Complex Sinusoids: FFT

$$\begin{bmatrix} \vec{X}[0] \\ \vec{X}[1] \\ \vec{X}[2] \\ \vec{X}[3] \end{bmatrix} = \begin{bmatrix} | & | & | & | \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \\ x[4] \\ x[6] \end{bmatrix} + \begin{bmatrix} e^{-2\pi i(0)/8} & & & \\ & e^{-2\pi i(1)/8} & & \\ & & e^{-2\pi i(2)/8} & \\ & & & e^{-2\pi i(3)/8} \end{bmatrix} \begin{bmatrix} | & | & | & | \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \\ x[5] \\ x[7] \end{bmatrix}$$

$$\begin{bmatrix} \vec{X}[4] \\ \vec{X}[5] \\ \vec{X}[6] \\ \vec{X}[7] \end{bmatrix} = \begin{bmatrix} | & | & | & | \end{bmatrix} \begin{bmatrix} \vec{x}[0] \\ \vec{x}[2] \\ \vec{x}[4] \\ \vec{x}[6] \end{bmatrix} + \begin{bmatrix} e^{-2\pi i(4)/8} & & & \\ & e^{-2\pi i(5)/8} & & \\ & & e^{-2\pi i(6)/8} & \\ & & & e^{-2\pi i(7)/8} \end{bmatrix} \begin{bmatrix} | & | & | & | \end{bmatrix} \begin{bmatrix} \vec{x}[1] \\ \vec{x}[3] \\ \vec{x}[5] \\ \vec{x}[7] \end{bmatrix}$$

Discrete Complex Sinusoids: FFT

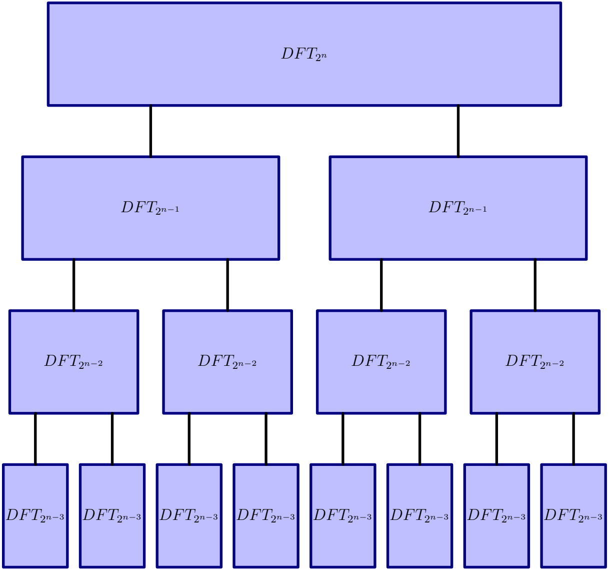
If $n = 1$, then set $\text{DFT}^1(\vec{x}) := \vec{x}$, otherwise apply the following steps:

1. Compute $\text{DFT}^{[n/2]}(\vec{x}_{\text{even}})$.
2. Compute $\text{DFT}^{[n/2]}(\vec{x}_{\text{odd}})$.
3. For $k = 1, 2, \ldots, n/2$ set

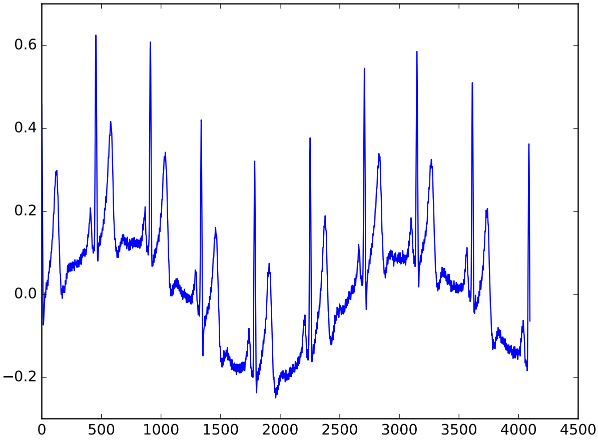
$$\text{DFT}^{[n]}(\vec{x})_k := \text{DFT}^{[n/2]}(\vec{x}_{\text{even}})_k + \exp\left(-\frac{i2\pi k}{n}\right)\text{DFT}^{[n/2]}(\vec{x}_{\text{odd}})_k,$$

$$\text{DFT}^{[n]}(\vec{x})_{k+n/2} := \text{DFT}^{[n/2]}(\vec{x}_{\text{even}})_k - \exp\left(-\frac{i2\pi k}{n}\right)\text{DFT}^{[n/2]}(\vec{x}_{\text{odd}})_k.$$

Discrete Complex Sinusoids: FFT



Discrete Complex Sinusoids: ECG



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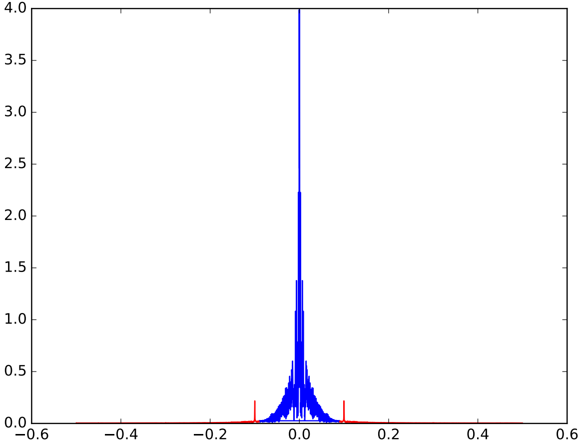

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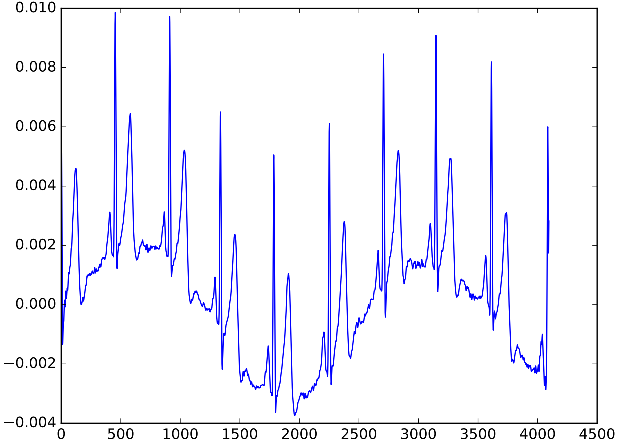
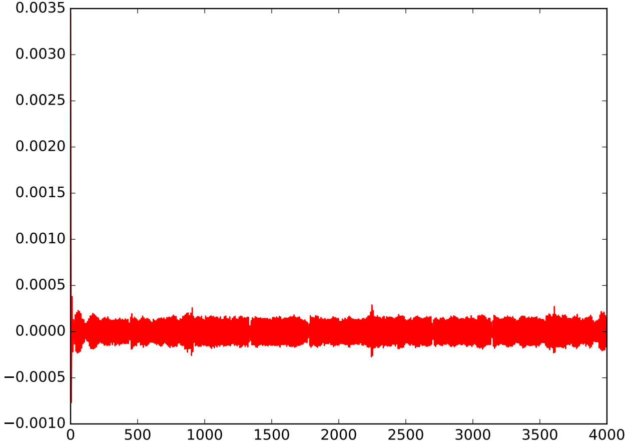
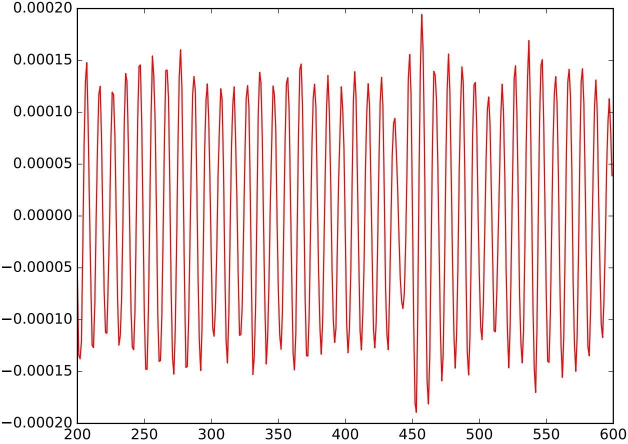


Discrete Complex Sinusoids: ECG



Discrete Complex Sinusoids: ECG



Two-dimensional DFT

The 2D DFT $\widehat{M}$ of an image $M \in \mathbb{C}^{n \times n}$ is given by

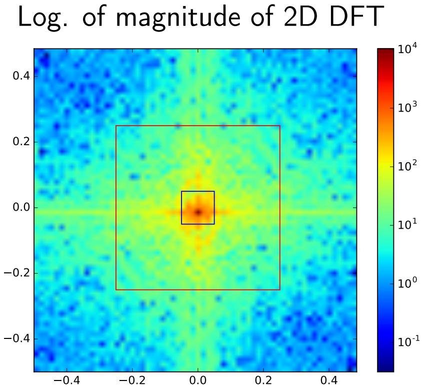
$$\widehat{M}[k_1, k_2] := \left\langle M, \vec{h}^{\,2D}_{k_1,k_2} \right\rangle, \qquad 0 \le k_1, k_2 \le n-1$$

$$\vec{h}^{\,2D}_{k_1,k_2} := \vec{h}^{\,[n]}_{k_1} \left(\vec{h}^{\,[n]}_{k_2} \right)^T$$

$$= \begin{bmatrix} 1 & e^{\frac{i2\pi k_2}{n}} & \cdots & e^{\frac{i2\pi k_2 (n-1)}{n}} \\ e^{\frac{i2\pi k_1}{n}} & e^{\frac{i2\pi \left(k_1 + k_2\right)}{n}} & \cdots & e^{\frac{i2\pi \left(k_1 + k_2 (n-1)\right)}{n}} \\ & & \cdots & \\ e^{\frac{i2\pi k_1 (n-1)}{n}} & e^{\frac{i2\pi \left(k_1 (n-1) + k_2\right)}{n}} & \cdots & e^{\frac{i2\pi \left(k_1 (n-1) + k_2 (n-1)\right)}{n}} \end{bmatrix}$$

$$\widehat{M} = WMW$$

Two-dimensional DFT



Two-dimensional DFT

Low-pass component

Band-pass component

High-pass component

Discrete cosine transform (DCT)

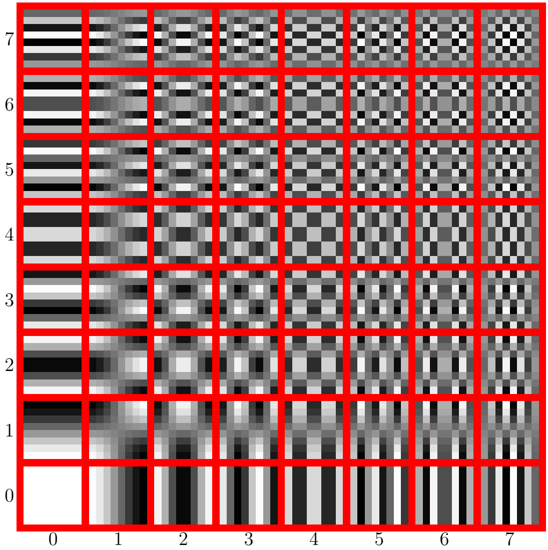
Variant of DFT for real signals

Signal is interpreted as one half of a symmetric signal

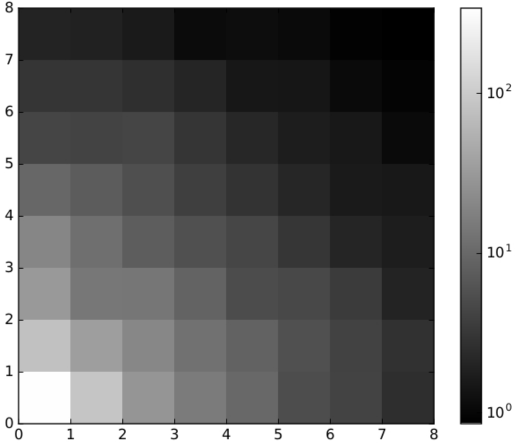
DFT then only involves cosines

Very important in image processing: low-frequency DCT components contain most of the energy in natural images

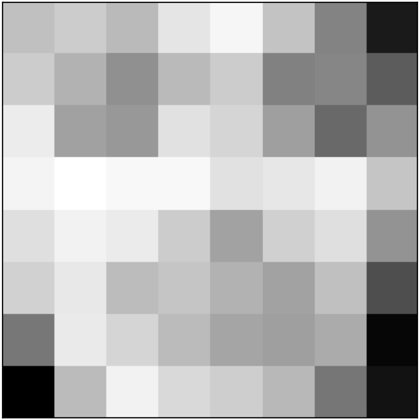
8 × 8 DCT basis vectors



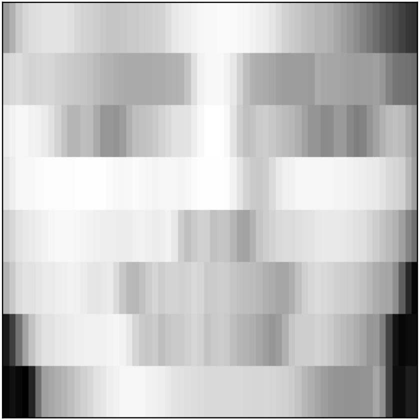
Average magnitudes of each 2D DCT coefficient in a database of patches



Projection of 8×8 patches onto first DCT basis vector



Projection of 8×8 patches onto first 5 DCT basis vectors



Projection of 8×8 patches onto first 15 DCT basis vectors

Projection of 8×8 patches onto first 30 DCT basis vectors

Projection of 8×8 patches onto first 50 DCT basis vectors

Original image

Quantizing the low frequencies

Quantizing the high frequencies

JPEG algorithm

1. Choose a quality setting $Q \in (0, 100)$
2. Divide image into 8×8 pixel patches
3. Compute the 2D DCT of each patch
4. Let $\widehat{P} \in \mathbb{R}^{8\times 8}$ denote the 2D DCT of a patch. Set

$$\widehat{P}'_{ij} = \text{round}\left(\frac{\widehat{P}_{ij}}{S(Q)M_{ij}}\right) S(Q)M_{ij}, \tag{1}$$

   where $S(Q)$ is the quality scaling factor:

$$S(Q) := \begin{cases} \frac{100-Q}{50} & \text{if } Q > 50 \\ \frac{50}{Q} & \text{otherwise} \end{cases} \tag{2}$$

5. Compute the inverse 2D DCT of each quantized patch $\widehat{P}'$ and encode

JPEG DCT Quantization Matrix

$$M = \begin{bmatrix} 16 & 11 & 10 & 16 & 24 & 40 & 51 & 61 \\ 12 & 12 & 14 & 19 & 26 & 58 & 60 & 55 \\ 14 & 13 & 16 & 24 & 40 & 57 & 69 & 56 \\ 14 & 17 & 22 & 29 & 51 & 87 & 80 & 62 \\ 18 & 22 & 37 & 56 & 68 & 109 & 103 & 77 \\ 24 & 35 & 55 & 64 & 81 & 104 & 113 & 92 \\ 49 & 64 & 78 & 87 & 103 & 121 & 120 & 101 \\ 72 & 92 & 95 & 98 & 112 & 100 & 103 & 99 \end{bmatrix}$$

JPEG

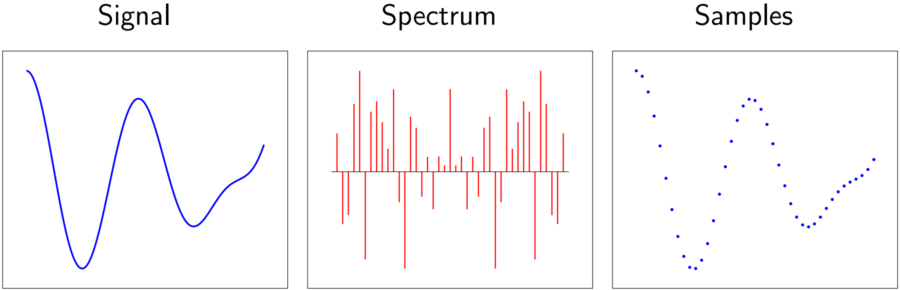
Fourier Representations

Sampling theorem

Convolution

Wiener deconvolution

Sampling a bandlimited signal



Important questions

1. What sampling rate is necessary to preserve all the information?
2. How can we reconstruct the signal from the samples?

Nyquist-Shannon-Kotelnikov sampling theorem

Any bandlimited signal $g \in \mathcal{L}_2\,[0,1]$ of the form

$$g\,(t) := \sum_{k=-k_c}^{k_c} G\,[k]\,h_k\,(t)$$

can be recovered from n samples $g\,(0)$, $g\,(1/n)$, $\ldots$, $g\,((n-1)/n)$ as long as
the sampling rate $f_s := n$ satisfies

$$f_s \geq 2k_c + 1$$

which is known as the Nyquist rate

Proof

$$\vec{g}_n := \begin{bmatrix} g(0) \\ g\left(\frac{1}{n}\right) \\ \cdots \\ g\left(\frac{n-1}{n}\right) \end{bmatrix} = \begin{bmatrix} \sum_{k=-k_c}^{k_c} G[k]\, h_k(0) \\ \sum_{k=-k_c}^{k_c} G[k]\, h_k\left(\frac{1}{n}\right) \\ \cdots \\ \sum_{k=-k_c}^{k_c} G[k]\, h_k\left(\frac{n-1}{n}\right) \end{bmatrix}$$

Proof

$$\vec{g}_n := \begin{bmatrix} g(0) \\ g\left(\frac{1}{n}\right) \\ \dots \\ g\left(\frac{n-1}{n}\right) \end{bmatrix} = \begin{bmatrix} \sum_{k=-k_c}^{k_c} G[k]\, h_k(0) \\ \sum_{k=-k_c}^{k_c} G[k]\, h_k\left(\frac{1}{n}\right) \\ \dots \\ \sum_{k=-k_c}^{k_c} G[k]\, h_k\left(\frac{n-1}{n}\right) \end{bmatrix}$$

$$= \sum_{k=-k_c}^{k_c} G[k] \begin{bmatrix} h_k\left(\frac{1}{n}\right) \\ h_k\left(\frac{2}{n}\right) \\ \dots \\ h_k(1) \end{bmatrix} = \sum_{k=-k_c}^{k_c} G[k]\, \vec{h}_k^{[n]}$$

Proof

$$\begin{aligned}\vec{g}_n &= \begin{bmatrix}\vec{h}^{[n]}_{-k_s} & \vec{h}^{[n]}_{-k_s+1} & \cdots & \vec{h}^{[n]}_{k_s}\end{bmatrix} \vec{G} \\ &= F\vec{G}\end{aligned}$$

$$\vec{G}\,[k] := \begin{cases} G\,[k]\,, & \text{if } |k| \leq k_c \\ 0, & \text{otherwise} \end{cases}$$

F is a square matrix with orthogonal columns

Dirichlet-kernel interpolation

Any bandlimited signal $g \in \mathcal{L}_2[0,1]$ of the form

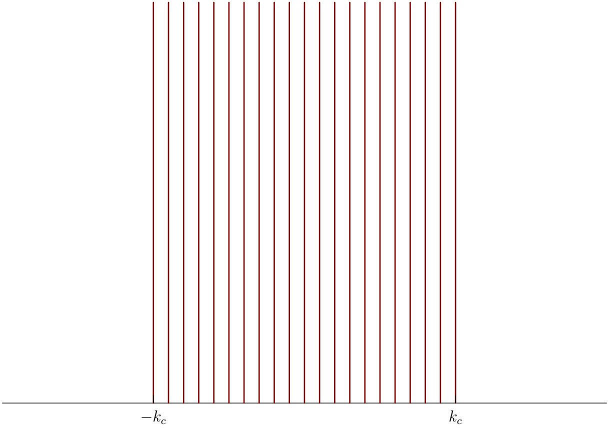
$$g(t) := \sum_{k=-k_c}^{k_c} G[k]\, h_k(t)$$

satisfies

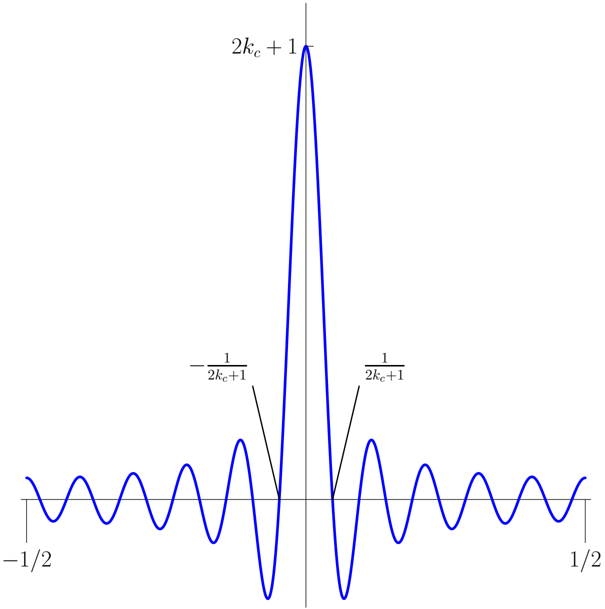
$$g(t) = \frac{1}{n} \sum_{j=0}^{n-1} g(j/n)\, d(t - j/n)$$

where d is a Dirichlet kernel with cut-off frequency k_c

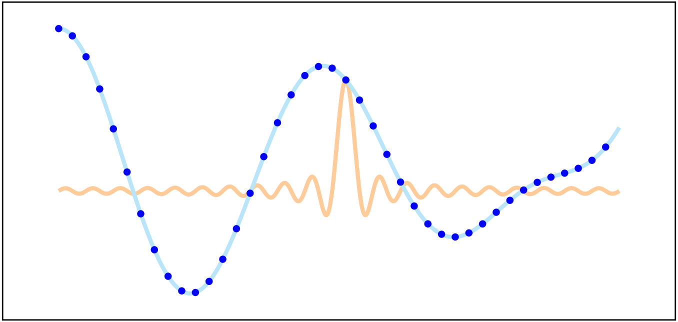
Dirichlet kernel (spectrum)



Dirichlet kernel



Dirichlet-kernel interpolation



Proof

$$\vec{a}_t := \begin{bmatrix} \exp\left(-2\pi k_s t\right) & \exp\left(-2\pi(k_s - 1)t\right) & \cdots & \exp\left(2\pi k_s t\right) \end{bmatrix}^T$$

$$F^* = \begin{bmatrix} \vec{a}_0 & \vec{a}_{1/n} & \cdots & \vec{a}_{(n-1)/n} \end{bmatrix}$$

Proof

$$\vec{a}_t := \begin{bmatrix} \exp\left(-2\pi k_s t\right) & \exp\left(-2\pi (k_s - 1) t\right) & \cdots & \exp\left(2\pi k_s t\right) \end{bmatrix}^T$$

$$F^* = \begin{bmatrix} \vec{a}_0 & \vec{a}_{1/n} & \cdots & \vec{a}_{(n-1)/n} \end{bmatrix}$$

$$\vec{G} = F^* \vec{g}_n = \frac{1}{n} \sum_{j=1}^{n} g\left(j/n\right) \vec{a}_{j/n}$$

Proof

$$g\left(t\right)=\frac{1}{n}\sum_{k=-k_c}^{k_c}G\left[k\right]e^{-i2\pi kt}=\frac{1}{n}\vec{a}_t^{*}\vec{G}$$

Proof

$$g(t) = \frac{1}{n} \sum_{k=-k_c}^{k_c} G[k]\, e^{-i2\pi kt} = \frac{1}{n} \vec{a}_t^{\,*} \vec{G}$$

$$d(t-\tau) = \sum_{k=-k_c}^{k_c} e^{-i2\pi k(t-\tau)} = \vec{a}_t^{\,*} \vec{a}_\tau$$

Proof

$$\vec{G} = \frac{1}{n} \sum_{j=0}^{n-1} g\left(j/n\right) \vec{a}_{j/n} \qquad g\left(t\right) = \frac{1}{n} \vec{a}_t^* \vec{G} \qquad d\left(t - \tau\right) = \vec{a}_t^* \vec{a}_\tau$$

$$g\left(t\right) = \frac{1}{n} \vec{a}_t^* \vec{G}$$

Proof

$$\vec{G} = \frac{1}{n}\sum_{j=0}^{n-1} g\left(j/n\right)\vec{a}_{j/n} \qquad g\left(t\right) = \frac{1}{n}\vec{a}_t^*\vec{G} \qquad d\left(t-\tau\right) = \vec{a}_t^*\vec{a}_\tau$$

$$\begin{aligned} g\left(t\right) &= \frac{1}{n}\vec{a}_t^*\vec{G} \\ &= \frac{1}{n}\sum_{j=0}^{n-1} g\left(j/n\right)\vec{a}_t^*\vec{a}_{j/n} \end{aligned}$$

Proof

$$\vec{G} = \frac{1}{n}\sum_{j=0}^{n-1} g\left(j/n\right)\vec{a}_{j/n} \qquad g\left(t\right) = \frac{1}{n}\vec{a}_t^{*}\vec{G} \qquad d\left(t-\tau\right) = \vec{a}_t^{*}\vec{a}_\tau$$

$$\begin{aligned} g\left(t\right) &= \frac{1}{n}\vec{a}_t^{*}\vec{G} \\ &= \frac{1}{n}\sum_{j=0}^{n-1} g\left(j/n\right)\vec{a}_t^{*}\vec{a}_{j/n} \\ &= \frac{1}{n}\sum_{j=0}^{n-1} g\left(j/n\right) d\left(t-j/n\right) \end{aligned}$$

Aliasing

We sample

$$g(t) := \sum_{k=-k_c}^{k_c+1} G[k] \, h_k(t)$$

at a rate $k_s := k_c/2$ (instead of $k_c + 1$), so $n := 2k_s + 1 = k_c + 1$

Aliasing

$$\vec{g}_n := \begin{bmatrix} g\left(0\right) \\ g\left(\frac{1}{n}\right) \\ \cdots \\ g\left(\frac{n-1}{n}\right) \end{bmatrix} = \begin{bmatrix} \sum_{k=-k_c}^{k_c} G\left[k\right] h_k\left(0\right) \\ \sum_{k=-k_c}^{k_c} G\left[k\right] h_k\left(\frac{1}{n}\right) \\ \cdots \\ \sum_{k=-k_c}^{k_c} G\left[k\right] h_k\left(\frac{n-1}{n}\right) \end{bmatrix}$$

$$= \sum_{k=-k_c}^{k_c} G\left[k\right] \begin{bmatrix} h_k\left(0\right) \\ h_k\left(\frac{1}{n}\right) \\ \cdots \\ h_k\left(\frac{n-1}{n}\right) \end{bmatrix} = \sum_{k=-k_c}^{k_c} G\left[k\right] \vec{h}_k^{[n]}$$

Aliasing

For any k, $\vec{h}_k^{[n]} = \vec{h}_{k+n}^{[n]}$

$$\begin{aligned}\vec{g}_n &= \begin{bmatrix} \vec{h}_{-k_c}^{[n]} & \cdots & \vec{h}_0^{[n]} & \vec{h}_1^{[n]} & \cdots & \vec{h}_{k_c+1}^{[n]} \end{bmatrix} \vec{G} \\ &= \begin{bmatrix} \vec{h}_{-2k_s}^{[n]} & \cdots & \vec{h}_0^{[n]} & \vec{h}_1^{[n]} & \cdots & \vec{h}_{2k_s+1}^{[n]} \end{bmatrix} \vec{G}\end{aligned}$$

$\widetilde{F}$ is a square matrix with orthogonal columns

Aliasing

For any k, $\vec{h}_k^{[n]} = \vec{h}_{k+n}^{[n]}$

$$\begin{aligned}
\vec{g}_n &= \begin{bmatrix} \vec{h}_{-k_c}^{[n]} & \cdots & \vec{h}_0^{[n]} & \vec{h}_1^{[n]} & \cdots & \vec{h}_{k_c+1}^{[n]} \end{bmatrix} \vec{G} \\
&= \begin{bmatrix} \vec{h}_{-2k_s}^{[n]} & \cdots & \vec{h}_0^{[n]} & \vec{h}_1^{[n]} & \cdots & \vec{h}_{2k_s+1}^{[n]} \end{bmatrix} \vec{G} \\
&= \begin{bmatrix} \vec{h}_1^{[n]} & \cdots & \vec{h}_n^{[n]} & \vec{h}_1^{[n]} & \cdots & \vec{h}_n^{[n]} \end{bmatrix} \vec{G} \\
&= \begin{bmatrix} \widetilde{F} & \widetilde{F} \end{bmatrix} \begin{bmatrix} \vec{G}_1 \\ \vec{G}_2 \end{bmatrix}
\end{aligned}$$

$\widetilde{F}$ is a square matrix with orthogonal columns

Aliasing

If $n = 2k_c + 1$

$$\vec{G} = \frac{1}{n}\widetilde{F}^* \vec{g}_n$$

Aliasing

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In this case

$$\begin{aligned}\vec{G}_{\text{aliased}} &= \frac{1}{n}\widetilde{F}^*\vec{g}_n \\ &= \frac{1}{n}\widetilde{F}^*\begin{bmatrix}\widetilde{F} & \widetilde{F}\end{bmatrix}\begin{bmatrix}\vec{G}_1 \\ \vec{G}_2\end{bmatrix}\end{aligned}$$

Aliasing

If $n = 2k_c + 1$

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In this case

$$\begin{aligned}
\vec{G}_{\text{aliased}} &= \frac{1}{n}\widetilde{F}^*\vec{g}_n \\
&= \frac{1}{n}\widetilde{F}^*\begin{bmatrix}\widetilde{F} & \widetilde{F}\end{bmatrix}\begin{bmatrix}\vec{G}_1 \\ \vec{G}_2\end{bmatrix} \\
&= \vec{G}_1 + \vec{G}_2
\end{aligned}$$

Fourier Representations

Sampling theorem

Convolution

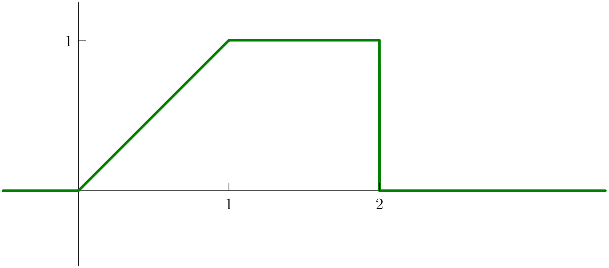
Wiener deconvolution

Convolution

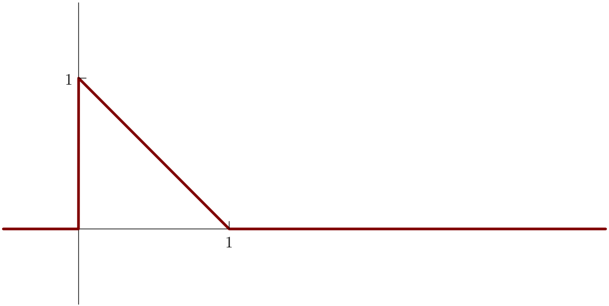
The convolution of two functions $f, g \in \mathcal{L}_2[-1/2, 1/2]$ is defined as

$$f * g(t) := \int_{-1/2}^{1/2} f(u)\, g(t-u)\, \mathrm{d}u$$

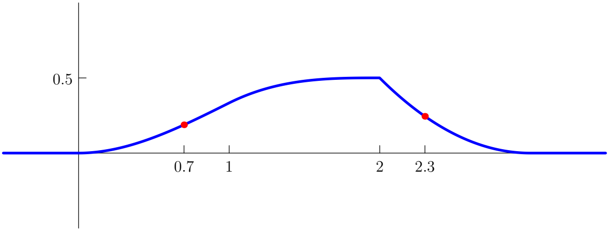
f



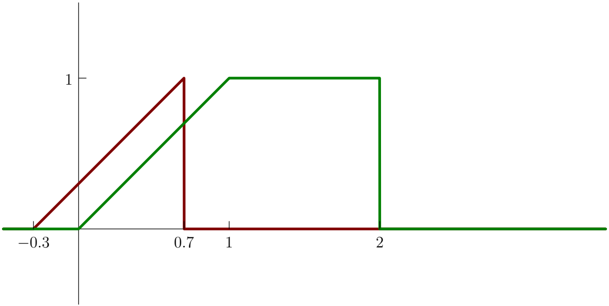
g



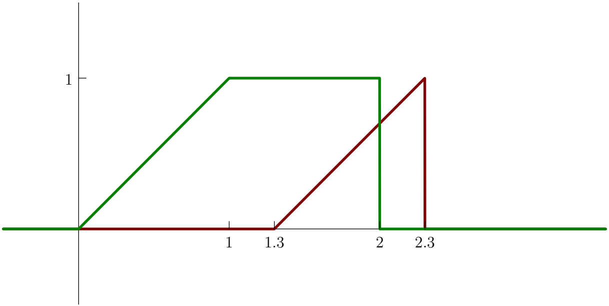
$f * g$



$f\,(t)\,g\,(0.7 - t)$



$f(t)\,g(2.3 - t)$



Time shift

The τ-shifted version of a function $f \in \mathcal{L}_2 [-1/2, 1/2]$ is

$$f_{[\tau]}(t) := f(t - \tau)$$

where the shift is *circular* (it wraps around)

For any shift τ

$$F_{[\tau]}[k] = \exp(-i2\pi k\tau) F[k]$$

Proof

We interpret f as a periodic function such that $f(t+1) = f(t)$

$$F_{[\tau]}[k] = \int_{-1/2}^{1/2} f(t-\tau)\exp(-i2\pi kt)\,\mathrm{d}t$$

Proof

We interpret f as a periodic function such that $f(t+1) = f(t)$

$$
\begin{aligned}
F_{[\tau]}[k] &= \int_{-1/2}^{1/2} f(t-\tau) \exp(-i2\pi kt)\, \mathrm{d}t \\
&= \int_{-1/2-\tau}^{1/2-\tau} f(u) \exp(-i2\pi k(u+\tau))\, \mathrm{d}t
\end{aligned}
$$

Proof

We interpret f as a periodic function such that $f(t+1) = f(t)$

$$\begin{aligned}
F_{[\tau]}[k] &= \int_{-1/2}^{1/2} f(t-\tau) \exp(-i2\pi kt)\, \mathrm{d}t \\
&= \int_{-1/2-\tau}^{1/2-\tau} f(u) \exp(-i2\pi k(u+\tau))\, \mathrm{d}t \\
&= \exp(-i2\pi k\tau) F[k]
\end{aligned}$$

Convolution in time is multiplication in frequency

Let $r := f * g$ for $f, g \in \mathcal{L}_2[-1/2, 1/2]$. Then

$$R[k] = F[k]\, G[k]$$

We can compute convolutions very fast using the FFT

Proof

Let $r := f * g$ for $f, g \in \mathcal{L}_2 [-1/2, 1/2]$. Then

$$R[k] := \int_{-1/2}^{1/2} \exp(-i2\pi kt) \, f * g(t) \, \mathrm{d}t$$

Proof

Let $r := f * g$ for $f, g \in \mathcal{L}_2\left[-1/2, 1/2\right]$. Then

$$
\begin{aligned}
R\left[k\right] &:= \int_{-1/2}^{1/2} \exp\left(-i2\pi kt\right) f * g\left(t\right)\, \text{d}t \\
&= \int_{-1/2}^{1/2} \exp\left(-i2\pi kt\right) \int_{-1/2}^{1/2} f\left(u\right) g\left(t-u\right) \text{d}u \, \text{d}t
\end{aligned}
$$

Proof

Let $r := f * g$ for $f, g \in \mathcal{L}_2[-1/2, 1/2]$. Then

$$
\begin{aligned}
R[k] &:= \int_{-1/2}^{1/2} \exp(-i2\pi kt) f * g(t) \, \mathrm{d}t \\
&= \int_{-1/2}^{1/2} \exp(-i2\pi kt) \int_{-1/2}^{1/2} f(u) g(t-u) \, \mathrm{d}u \, \mathrm{d}t \\
&= \int_{-1/2}^{1/2} f(u) \int_{-1/2}^{1/2} \exp(-i2\pi kt) g(t-u) \, \mathrm{d}t \, \mathrm{d}u
\end{aligned}
$$

Proof

Let $r := f * g$ for $f, g \in \mathcal{L}_2\left[-1/2, 1/2\right]$. Then

$$\begin{aligned}
R\left[k\right] &:= \int_{-1/2}^{1/2} \exp\left(-i2\pi kt\right) f * g\left(t\right)\, \mathrm{d}t \\
&= \int_{-1/2}^{1/2} \exp\left(-i2\pi kt\right) \int_{-1/2}^{1/2} f\left(u\right) g\left(t-u\right)\, \mathrm{d}u\, \mathrm{d}t \\
&= \int_{-1/2}^{1/2} f\left(u\right) \int_{-1/2}^{1/2} \exp\left(-i2\pi kt\right) g\left(t-u\right)\, \mathrm{d}t\, \mathrm{d}u \\
&= \int_{-1/2}^{1/2} f\left(u\right) G\left[k\right] \exp\left(-i2\pi ku\right)\, \mathrm{d}t\, \mathrm{d}u
\end{aligned}$$

Proof

Let $r := f * g$ for $f, g \in \mathcal{L}_2[-1/2, 1/2]$. Then

$$
\begin{aligned}
R[k] &:= \int_{-1/2}^{1/2} \exp(-i2\pi kt)\, f * g(t)\, \mathrm{d}t \\
&= \int_{-1/2}^{1/2} \exp(-i2\pi kt) \int_{-1/2}^{1/2} f(u)\, g(t-u)\, \mathrm{d}u\, \mathrm{d}t \\
&= \int_{-1/2}^{1/2} f(u) \int_{-1/2}^{1/2} \exp(-i2\pi kt)\, g(t-u)\, \mathrm{d}t\, \mathrm{d}u \\
&= \int_{-1/2}^{1/2} f(u)\, G[k] \exp(-i2\pi ku)\, \mathrm{d}t\, \mathrm{d}u \\
&= F[k]\, G[k]
\end{aligned}
$$

Central limit theorem

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of iid random variables with mean μ and bounded variance σ^2

The sequence of averages $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots$ is defined as

$$\mathbf{a}_i := \frac{1}{i} \sum_{j=1}^{i} \mathbf{x}_j$$

Central limit theorem

The sequence $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, ...

$$\mathbf{b}_i := \sqrt{i}(\mathbf{a}_i - \mu)$$

converges in distribution to a Gaussian random variable with mean 0 and variance σ^2

For any $x \in \mathbb{R}$

$$\lim_{i \to \infty} f_{\mathbf{b}_i}(x) := \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

For large i the theorem suggests that the average $\mathbf{a}_i$ is approximately Gaussian with mean μ and variance $\sigma/\sqrt{n}$

Sum of independent random variables

If $\mathbf{x}$ and $\mathbf{y}$ are independent random variables, the pdf of

$$\mathbf{z} = \mathbf{x} + \mathbf{y}$$

is equal to the convolution of $f_{\mathbf{x}}$ and $f_{\mathbf{y}}$

$$f_{\mathbf{z}}(z) = \int_{u=-\infty}^{\infty} f_{\mathbf{x}}(z-u)\, f_{\mathbf{y}}(u)\, \mathrm{d}u$$

Proof

$$F_{\mathbf{z}}(z)$$

Proof

$$F_{\mathbf{z}}(z) = \mathrm{P}\left(\mathbf{x} + \mathbf{y} \leq z\right)$$

Proof

$$\begin{aligned} F_{\mathbf{z}}(z) &= \mathrm{P}\left(\mathbf{x} + \mathbf{y} \leq z\right) \\ &= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} f_{\mathbf{x},\mathbf{y}}(x, y)\, \mathrm{d}x\, \mathrm{d}y \end{aligned}$$

Proof

$$
\begin{aligned}
F_{\mathbf{z}}(z) &= \mathrm{P}\left(\mathbf{x}+\mathbf{y} \leq z\right) \\
&= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} f_{\mathbf{x},\mathbf{y}}(x, y)\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{y=-\infty}^{\infty} \int_{u=-\infty}^{z} f_{\mathbf{x},\mathbf{y}}(u-y, y)\, \mathrm{d}u\, \mathrm{d}y \qquad (u = x+y)
\end{aligned}
$$

Proof

$$
\begin{aligned}
F_{\mathbf{z}}(z) &= \mathrm{P}\left(\mathbf{x}+\mathbf{y} \leq z\right) \\
&= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} f_{\mathbf{x},\mathbf{y}}(x, y)\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{y=-\infty}^{\infty} \int_{u=-\infty}^{z} f_{\mathbf{x},\mathbf{y}}(u-y, y)\, \mathrm{d}u\, \mathrm{d}y \qquad (u = x + y) \\
&= \int_{u=-\infty}^{z} \int_{y=-\infty}^{\infty} f_{\mathbf{x},\mathbf{y}}(u-y, y)\, \mathrm{d}y\, \mathrm{d}u
\end{aligned}
$$

Proof

$$
\begin{aligned}
F_{\mathbf{z}}(z) &= \mathrm{P}\left(\mathbf{x} + \mathbf{y} \leq z\right) \\
&= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} f_{\mathbf{x},\mathbf{y}}(x, y)\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{y=-\infty}^{\infty} \int_{u=-\infty}^{z} f_{\mathbf{x},\mathbf{y}}(u - y, y)\, \mathrm{d}u\, \mathrm{d}y \qquad (u = x + y) \\
&= \int_{u=-\infty}^{z} \int_{y=-\infty}^{\infty} f_{\mathbf{x},\mathbf{y}}(u - y, y)\, \mathrm{d}y\, \mathrm{d}u
\end{aligned}
$$

$$
f_{\mathbf{z}}(z) = \int_{y=-\infty}^{\infty} f_{\mathbf{x},\mathbf{y}}(z - y, y)\, \mathrm{d}y
$$

Proof

$$\begin{aligned}
F_{\mathbf{z}}(z) &= \mathrm{P}\left(\mathbf{x} + \mathbf{y} \leq z\right) \\
&= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} f_{\mathbf{x},\mathbf{y}}(x, y) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{y=-\infty}^{\infty} \int_{u=-\infty}^{z} f_{\mathbf{x},\mathbf{y}}(u - y, y) \, \mathrm{d}u \, \mathrm{d}y \qquad (u = x + y) \\
&= \int_{u=-\infty}^{z} \int_{y=-\infty}^{\infty} f_{\mathbf{x},\mathbf{y}}(u - y, y) \, \mathrm{d}y \, \mathrm{d}u
\end{aligned}$$

$$\begin{aligned}
f_{\mathbf{z}}(z) &= \int_{y=-\infty}^{\infty} f_{\mathbf{x},\mathbf{y}}(z - y, y) \, \mathrm{d}y \\
&= \int_{y=-\infty}^{\infty} f_{\mathbf{x}}(z - y) f_{\mathbf{y}}(y) \, \mathrm{d}y
\end{aligned}$$

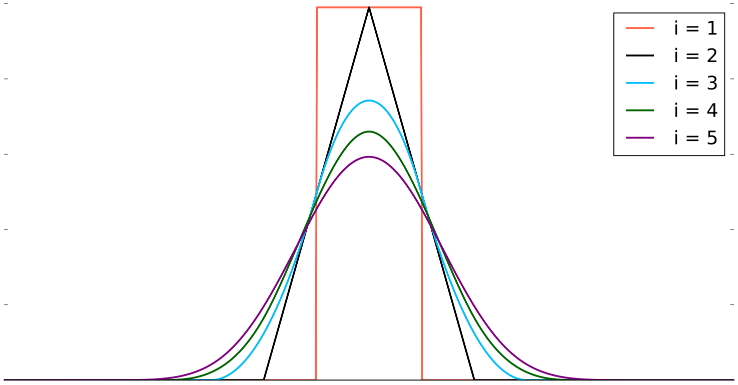
Sketch of proof of central limit theorem

Sequence of iid random variables $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ with pdf f

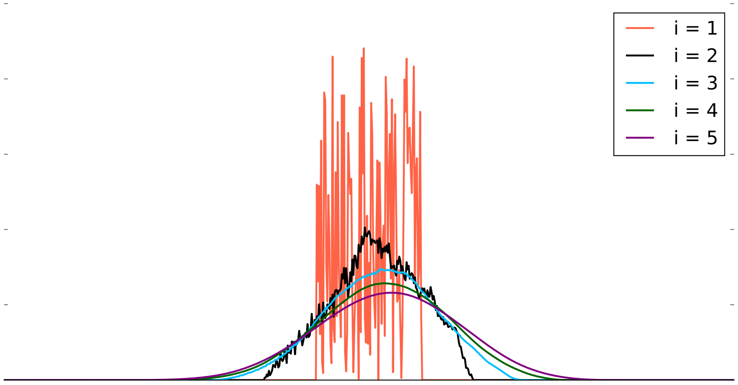
The pdf of the sum is given by

$$f_{\sum_{j=1}^{\infty} \mathbf{x}_j}(x) = (f * f * \cdots)(x)$$

Sketch of proof of central limit theorem



Sketch of proof of central limit theorem



Discrete convolution

The circular convolution of $\vec{x}, \vec{y} \in \mathbb{C}^n$ is

$$\vec{x} * \vec{y}\,[j] := \sum_{m=0}^{n-1} \vec{x}\,[m]\,\vec{y}\,[j-m]\,, \qquad 0 \leq j \leq n-1$$

where the shifts are circular, so that $\vec{x}\,[j] = \vec{x}\,[j+n]$ and $\vec{y}\,[j] = \vec{y}\,[j+n]$

Convolution matrix

$$C_{\vec{y}} := \begin{bmatrix} \vec{y}[0] & \vec{y}[n-1] & \cdots & \vec{y}[2] & \vec{y}[1] \\ \vec{y}[1] & \vec{y}[0] & \cdots & \vec{y}[3] & \vec{y}[2] \\ & & \cdots & & \\ \vec{y}[n-1] & \vec{y}[n-2] & \cdots & \vec{y}[1] & \vec{y}[0] \end{bmatrix}$$

Matrices with this structure are called circulant matrices

Assuming entries are numbered from 0 to $n-1$, for any $\vec{x} \in \mathbb{C}^n$

$$\vec{x} * \vec{y} = C_{\vec{y}} \vec{x}$$

Convolution in time is multiplication in frequency

Let $\vec{r} := \vec{x} * \vec{y}$ for $\vec{x}, \vec{y} \in \mathbb{C}^n$. Then

$$\vec{R}[k] = \vec{X}[k]\,\vec{Y}[k]$$

Discrete time shift

The m-shifted version of $\vec{x} \in \mathbb{C}^n$ is

$$\vec{x}_{[m]}[j] := \vec{x}(j - m)$$

where the shift is *circular*, so $\vec{x}(j + n) = \vec{x}(j)$

For any shift m we have

$$\vec{X}_{[m]}[k] = \exp(-i2\pi km)\,\vec{X}[k]$$

Proof

$$\vec{X}_{[m]}[k] = \sum_{j=0}^{n-1} \vec{x}_{[\tau]}[j] \exp(-2\pi kj)$$

Proof

$$\vec{X}_{[m]}[k] = \sum_{j=0}^{n-1} \vec{x}_{[\tau]}[j] \exp(-2\pi kj)$$
$$= \sum_{l=-m}^{n-1-m} \vec{x}[l] \exp(-2\pi k (l+m))$$

Proof

$$
\begin{aligned}
\vec{X}_{[m]}[k] &= \sum_{j=0}^{n-1} \vec{x}_{[\tau]}[j] \exp(-2\pi kj) \\
&= \sum_{l=-m}^{n-1-m} \vec{x}[l] \exp(-2\pi k(l+m)) \\
&= \exp(-i2\pi km) \vec{X}[k]
\end{aligned}
$$

Convolution in time is multiplication in frequency

$$R\left[k\right] := \sum_{j=0}^{n-1} \exp\left(-i2\pi kj\right) \sum_{m=0}^{n-1} \vec{x}\left[m\right] \vec{y}\left[j-m\right]$$

Convolution in time is multiplication in frequency

$$R[k] := \sum_{j=0}^{n-1} \exp(-i2\pi kj) \sum_{m=0}^{n-1} \vec{x}[m]\, \vec{y}[j-m]$$

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Convolution in time is multiplication in frequency

$$
\begin{aligned}
R[k] &:= \sum_{j=0}^{n-1} \exp(-i2\pi kj) \sum_{m=0}^{n-1} \vec{x}[m]\, \vec{y}[j-m] \\
&= \sum_{m=0}^{n-1} \vec{x}[m] \sum_{j=0}^{n-1} \exp(-i2\pi kj)\, \vec{y}[j-m] \\
&= \sum_{m=0}^{n-1} \vec{x}[m] \exp(-i2\pi km)\, \vec{Y}[k]
\end{aligned}
$$

Convolution in time is multiplication in frequency

$$
\begin{aligned}
R[k] &:= \sum_{j=0}^{n-1} \exp(-i2\pi kj) \sum_{m=0}^{n-1} \vec{x}[m]\, \vec{y}[j-m] \\
&= \sum_{m=0}^{n-1} \vec{x}[m] \sum_{j=0}^{n-1} \exp(-i2\pi kj)\, \vec{y}[j-m] \\
&= \sum_{m=0}^{n-1} \vec{x}[m] \exp(-i2\pi km)\, \vec{Y}[k] \\
&= \vec{X}[k]\, \vec{Y}[k]
\end{aligned}
$$

Eigendecomposition of circulant matrices

For any vector $\vec{x}$ $\Lambda_{\vec{Y}}$

$$
\begin{aligned}
C_{\vec{y}} &= \vec{x} * \vec{y} \\
&= \frac{1}{n} W^* \Lambda_{\vec{Y}} \vec{X} \\
&= \frac{1}{n} W^* \Lambda_{\vec{Y}} W \vec{x}.
\end{aligned}
$$

Eigendecomposition of circulant matrices

For any vector $\vec{x}$ $\Lambda_{\vec{Y}}$

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For any circulant matrix $C_{\vec{y}}$ corresponding to a vector $\vec{y}$

$$C_{\vec{y}} = \frac{1}{n} W^* \Lambda_{\vec{Y}} W$$

Fourier Representations

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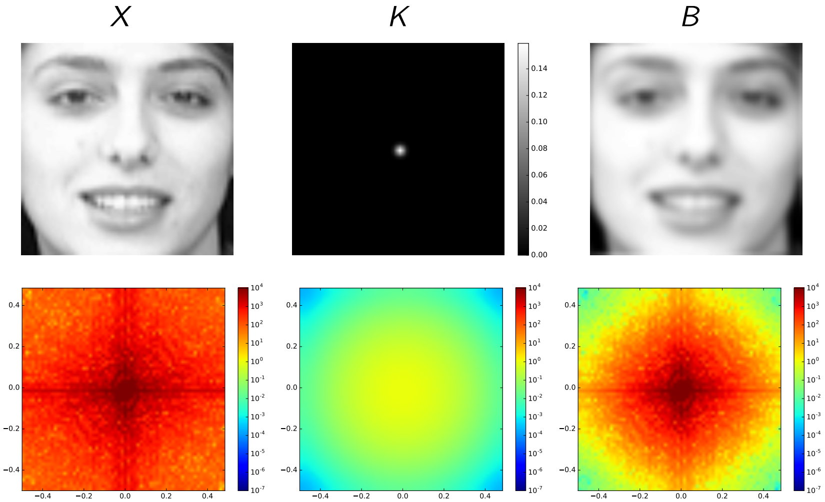
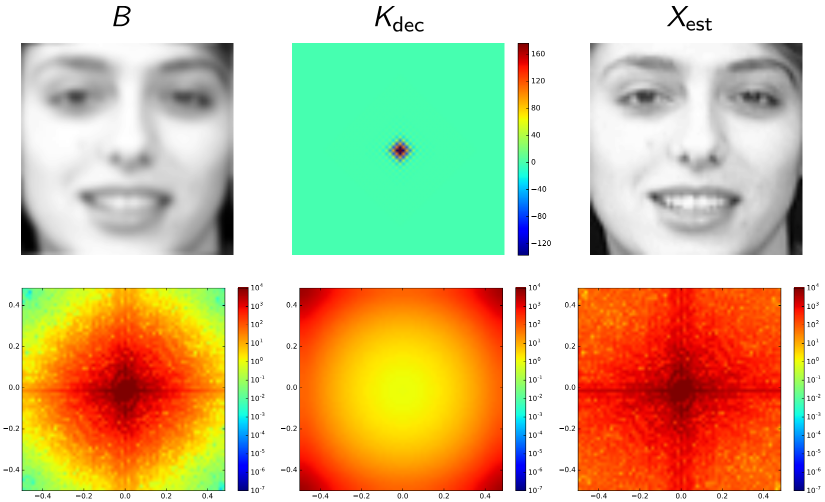
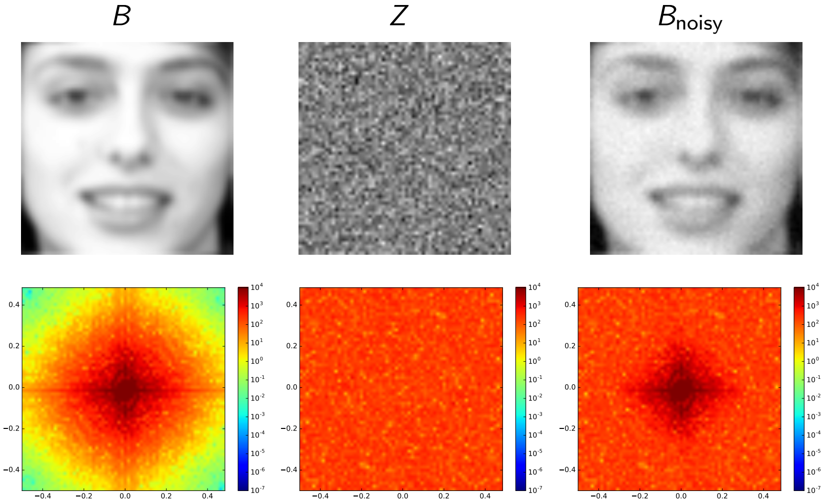
Noiseless data



Deconvolution

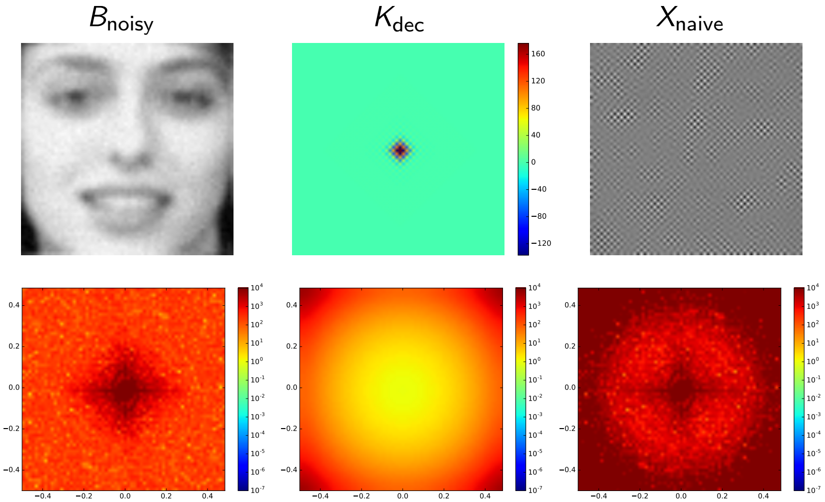


Noisy data



B
Z
B_noisy
−0.4 −0.2 0.0 0.2 0.4
10^4 10^3 10^2 10^1 10^0 10^-1 10^-2 10^-3 10^-4 10^-5 10^-6 10^-7

Naive deconvolution



Vector space of zero-mean random variables

Zero-mean complex-valued random variables form a vector space

The covariance is a valid inner product

The variance is the inner-product norm

By Chebyshev's inequality if $||\mathbf{x}||_{\langle \cdot, \cdot \rangle} = 0$, for any $\epsilon > 0$

$$\mathrm{P}\left(|\mathbf{x}| > \epsilon\right) \leq$$

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By Chebyshev's inequality if $||\mathsf{x}||_{\langle \cdot, \cdot \rangle} = 0$, for any $\epsilon > 0$

$$\mathrm{P}\left(|\mathsf{x}| > \epsilon\right) \leq \frac{\mathrm{Var}\left(\mathsf{x}\right)}{\epsilon^2} = 0$$

so x with probability one

Linear estimation

Let

$$\mathbf{y} = a\mathbf{x} + \mathbf{z}$$

what is the linear estimate

$$\mathbf{x}_{\text{MMSE}} := w\mathbf{y}$$

that minimizes

$$\mathrm{E}\left((\mathbf{x} - \mathbf{x}_{\text{MMSE}})^2\right)$$

Linear estimation

We want a vector in the span of $\mathbf{y}$ that minimizes

$$\mathrm{E}\left((\mathsf{x} - \mathsf{x}_{\mathsf{MMSE}})^2\right) = ||\mathsf{x} - \mathsf{x}_{\mathsf{MMSE}}||^2_{\langle\cdot,\cdot\rangle}$$

Projection

$$\mathcal{P}_{\mathsf{span}(\mathbf{y})}\,\mathbf{x} = \left\langle \mathbf{x}, \frac{\mathbf{y}}{||\mathbf{y}||_{\langle\cdot,\cdot\rangle}} \right\rangle \frac{\mathbf{y}}{||\mathbf{y}||_{\langle\cdot,\cdot\rangle}}$$

Projection

$$w = \frac{1}{||\mathbf{y}||_{\langle\cdot,\cdot\rangle}} \left\langle \mathbf{x}, \frac{\mathbf{y}}{||\mathbf{y}||_{\langle\cdot,\cdot\rangle}} \right\rangle$$

Projection

$$w = \frac{1}{||\mathbf{y}||_{\langle\cdot,\cdot\rangle}} \left\langle \mathbf{x}, \frac{\mathbf{y}}{||\mathbf{y}||_{\langle\cdot,\cdot\rangle}} \right\rangle$$

$$= \frac{\langle \mathbf{x}, a\mathbf{x} + \mathbf{z} \rangle}{||a\mathbf{x} + \mathbf{z}||^2_{\langle\cdot,\cdot\rangle}}$$

Projection

$$
\begin{aligned}
w &= \frac{1}{||\mathbf{y}||_{\langle\cdot,\cdot\rangle}} \left\langle \mathbf{x}, \frac{\mathbf{y}}{||\mathbf{y}||_{\langle\cdot,\cdot\rangle}} \right\rangle \\
&= \frac{\langle \mathbf{x}, a\mathbf{x} + \mathbf{z} \rangle}{||a\mathbf{x} + \mathbf{z}||^2_{\langle\cdot,\cdot\rangle}} \\
&= \frac{a\,||\mathbf{x}||^2_{\langle\cdot,\cdot\rangle} + a\,\langle \mathbf{x}, \mathbf{z} \rangle}{||a\mathbf{x} + \mathbf{z}||^2_{\langle\cdot,\cdot\rangle}}
\end{aligned}
$$

Projection

$$
\begin{aligned}
w &= \frac{1}{||\mathbf{y}||_{\langle\cdot,\cdot\rangle}} \left\langle \mathbf{x}, \frac{\mathbf{y}}{||\mathbf{y}||_{\langle\cdot,\cdot\rangle}} \right\rangle \\
&= \frac{\langle \mathbf{x}, a\mathbf{x} + \mathbf{z} \rangle}{||a\mathbf{x} + \mathbf{z}||^2_{\langle\cdot,\cdot\rangle}} \\
&= \frac{a\,||\mathbf{x}||^2_{\langle\cdot,\cdot\rangle} + a\,\langle \mathbf{x}, \mathbf{z} \rangle}{||a\mathbf{x} + \mathbf{z}||^2_{\langle\cdot,\cdot\rangle}} \\
&= \frac{a\,||\mathbf{x}||_{\langle\cdot,\cdot\rangle}}{a^2\,||\mathbf{x}||^2_{\langle\cdot,\cdot\rangle} + ||\mathbf{z}||^2_{\langle\cdot,\cdot\rangle}}
\end{aligned}
$$

Projection

$$
\begin{aligned}
w &= \frac{1}{||\mathbf{y}||_{\langle\cdot,\cdot\rangle}} \left\langle \mathbf{x}, \frac{\mathbf{y}}{||\mathbf{y}||_{\langle\cdot,\cdot\rangle}} \right\rangle \\
&= \frac{\langle \mathbf{x}, a\mathbf{x} + \mathbf{z} \rangle}{||a\mathbf{x} + \mathbf{z}||^2_{\langle\cdot,\cdot\rangle}} \\
&= \frac{a\,||\mathbf{x}||^2_{\langle\cdot,\cdot\rangle} + a\,\langle \mathbf{x}, \mathbf{z} \rangle}{||a\mathbf{x} + \mathbf{z}||^2_{\langle\cdot,\cdot\rangle}} \\
&= \frac{a\,||\mathbf{x}||_{\langle\cdot,\cdot\rangle}}{a^2\,||\mathbf{x}||^2_{\langle\cdot,\cdot\rangle} + ||\mathbf{z}||^2_{\langle\cdot,\cdot\rangle}} \\
&= \frac{a\sigma_{\mathbf{x}}}{a^2\sigma^2_{\mathbf{x}} + \sigma^2_{\mathbf{z}}}
\end{aligned}
$$

Wiener deconvolution

Given $B_{\text{noisy}} \in \mathbb{R}^{n \times n}$ and a kernel $K \in \mathbb{R}^{n \times n}$

1. Estimate variance of each 2D DFT coefficient $\sigma_Z[k_1, k_2]$.

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$$W[k_1, k_2] := \frac{\widehat{K}[k_1, k_2]\sigma_X[k_1, k_2]}{\widehat{K}[k_1, k_2]^2 \sigma_X[k_1, k_2]^2 + \sigma_Z[k_1, k_2]^2}$$

$$X_W := \mu_X[k_1, k_2] + W[k_1, k_2]\widehat{B}_c[k_1, k_2]$$

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$$X_W := \mu_X[k_1, k_2] + W[k_1, k_2]\widehat{B}_c[k_1, k_2]$$

5. Compute the inverse 2D DFT of X_W.

Wiener deconvolution