1. In one dimension, the Eulerian velocity is given to be $u(x,t) = 2x/(1+t)$. (a) Find the Lagrangian coordinate $x(a,t)$. (b) Find the Lagrangian velocity as a function of a,t. (c) Find the jacobian $\partial x/\partial a = J$ as a function of a,t. (d) If density satisfies $\rho(x,0) = x$ and mass is conserved, find $\rho(a,t)$ using the Lagrangian form of mass conservation. (e) From (a) and (d) evaluate ρ as a function of x,t, and verify that the Eulerian conservation of mass equation is satisfied by $\rho(x,t), u(x,t)$.

2. Consider the "point vortex " flow in two dimensions,

$$(u,v) = UL(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}),\ x^2+y^2 \neq 0,$$

where U, L are reference values of speed and length. (a) Show that the Lagrangian coordinates for this flow may be written

$$x(a,b,t) = R_0 \cos(\omega t + \theta_0),\ y(a,b,t) = R_0 \sin(\omega t + \theta_0)$$

where $R_0^2 = a^2 + b^2, \theta_0 = \arctan(b/a)$, and $\omega = UL/R_0^2$. (b) Consider, at $t = 0$ a small rectangle of marked fluid particles determined by the points $A(L,0), B(L+\Delta x, 0), C(L+\Delta x, \Delta y), D(L, \Delta y)$. If the points move with the fluid, once point A returns to its initial position what is the shape of the marked region? Since $(\Delta x, \Delta y)$ are small, you may assume the region remains a parallelogram. Do this, first, by computing the entry $\partial y/\partial a$ in the jacobian, evaluated at $A(L,0)$. Then verify your result by considering the "lag" of particle B as it moves on a slightly larger circle at a slightly slower speed, relative to particle A, for a time taken by A to complete one revolution.

3. As was noted in class, Lagrangian coordinates can use any unique labeling of fluid particles. To illustrate this, consider the Lagrangian coordinates in two dimensions

$$x(a,b,t) = a + \frac{1}{k} e^{kb} \sin k(a+ct),\ y = b - \frac{1}{k} e^{kb} \cos k(a+ct),$$

where k, c are constants. Note here a, b are *not* the intital coordinates. By examining the determinant of the Jacobian, veryfy that this gives a unique labeling of fluid particles. (These waves, which were discovered by Gerstner in 1802, represent gravity waves if $c^2 = g/k$ where g is the acceleration of gravity. They do not have any simple Eulerian representation.)

4. For the stagnation-point flow $(u,v) = U/L(x,-y)$, show that a fluid particle in the first quadrant which crosses the line $y = L$ at time $t = 0$, crosses the line $x = L$ at time $t = \frac{L}{U}\log(UL/\psi)$ on the streamline $Uxy/L = \psi$.(Hint: Consider a line integral of $\vec{u}\cdot\vec{ds}/(u^2+v^2)$ along a streamline.)

5. (Non-existence of large hummingbirds). Because of stress limitation on bones, it is known that the power available for the hovering of birds is proportion to L^2, where L is a typical length representing the size of the bird. Show that the power *required* for hovering is proportional to $L^{7/2}$. Assume bird weight proportional to L^3. Use the fact that the required power is the speed U of the downward jet created in hovering times the force needed to hover. Assume the downward jet area is proportional to L^2, and consider the momentum it carries.

6. Find a solution of the Navier-Stokes equations for a flow in two dimensions of the form $\vec{u} = (u(y,t), 0)$, $p = 0$, in the domaian bounded by $y = 0, H$. The surface $y = 0$ oscillates sinusoidally with amplitude A and frequency ω. The upper surface $y = H$ is held fixed. Because of the no-slip condition, we must have $u(0,t) = \omega A\cos(\omega t)$, and $u(H,t) = 0$. Show that $u(y,t)$ satisfies the heat equation $u_t - \nu u_{yy} = 0$ and solve for $u(y,t)$ by separation of variables. Compute the force (per unit area) on the two walls, and the momentum per unit area of the fluid, and verify Newton's law (force on fluid equals time derivative of total momentum, all per unit area).