Fluid dynamics of animal locomotion PROBLEM SET 2 Due October 6, 2004

1. Define the complex derivatives

$$\frac{d}{dz} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big),\ \frac{d}{d\bar z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big).$$

Applied to a complex-valued function $w = F(x, y) + iG(x, y)$, where f, g are very smooth, verify that
(a) $\frac{dw}{d\bar z} = 0 \Rightarrow\ \nabla^2 F = 0, \nabla^2 G = 0;$
(b) $\frac{d^2 w}{d\bar z^2} = 0 \Rightarrow\ \nabla^4 F = 0, \nabla^4 G = 0;$
Also show that $\frac{df}{d\bar z} = 0$ for any analytic function of the complex variable $z = x+iy$, and that $\frac{df(\bar z}{d\bar z}) = f'(\bar z)$. Thus show that general solutions of the biharmonic equation $\nabla^4\psi = 0$ in two dimensions are provided by the real and imaginary parts of $\bar z f(z) + g(z)$, where f, g are analytic functions of a complex variable. Relate this general result to the particular solutions $(A + By)e^{\pm kx})(\sin kx, \cos kx)$ used in the study of the swimming sheet.

2. What is wrong with the following reasoning? In the planar, stretching swimming sheet, the plane just oscillates back and forth, just like the the case of an oscillating plane wall (the Rayleigh problem discussed in class as an example of a solution of the Navier-Stokes equation). But in that problem the flow decayed exactly to zero exponentially in y. The same thing should happen for the "swimming" sheet, so in fact it cannot swim!

3. G.I. Taylor, in the paper distributed in class, found that an inextensible waving sheet swims with velocity

$$U \approx \frac{V}{2} b^2 k^2 (1 - \frac{19}{16} b^2 k^2),$$

where $V = \omega/k$.Taking $\frac{bk}{2\pi} = .07$, a value typical of a spermatozoan, show that the predicted swimming speed corresponds to about 13 oscillations of a particle of the sheet for each wavelength $2\pi/k$ of progress.

4.Verify that

$$\vec u = a^3 \frac{\vec r \times \Omega}{r^3} \vec r = (x, y, x), r = \sqrt{x^2 + y^2 + z^2},$$

where a is a positive contant and Ω is a constant 3-vector, is a solution of the Stokes equations in $r > a$ with zero pressure. Show that the flow field is that produced by a rigid sphere of radius a spinning with angular velocity Ω. Show that the torque on the sphere is $-8\pi a^3 \mu\Omega$, by integrating the stress component $\sigma_{r\theta}$ over the surface of ths sphere. (r, θ, ϕ) are spherical coordinates with axis Ω. (See attached sheet for NS equations in various coordinate systems.)

5. Using the same basic approach as for the uniqueness proof for the Stokes equations, prove the following result: Let a finite rigid 3D body move with steady velocity $\vec U$ in a fluid otherwise at rest, From the steady Navier-Stokes equations for a fluid of constant density, prove that

$$UD = \frac{\mu}{2} \int_V \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}^2 dV.$$

Here V is the domain exterior to the body. The quantity on the right is the total viscous dissipation in the fluid, so this is a mechanical energy equation, stating that the work done on the fluid by the body is egual to the rate of heating of the fluid by viscous dissipation. What conditions on the decay at infinity are needed in this proof?