ANSWERS – PROBLEM SET 3.

(1)(a) $e^{z-1} = 1+i$, $z = 1 + \log(1+i)$.

$\log(1+i) = \ln\sqrt{2} + i\frac{\pi}{4} + 2\pi n i$

$z = 1 + \ln\sqrt{2} + \pi i\left[\frac{1}{4} + 2n\right] \quad n = 0, \pm1, \pm2, \ldots$

(b) $e^f = e^u e^{iv} = e^u(\cos v + i \sin v)$

Since f is analytic on D and e^z is analytic everywhere, $e^{f(z)}$ is analytic on D. Hence the real and imaginary parts are conjugate harmonic, i.e. $U(x,y) = e^{u(x,y)}\cos v(x,y)$ and $V(x,y) = e^{u(x,y)}\sin v(x,y)$ are conjugate harmonic.

(2) (a) $\mathrm{Log}(1-i) = \ln\sqrt{2} - \frac{\pi}{4}i$ since $\mathrm{Arg}(1-i) = -\frac{\pi}{4}i$

$= \frac{1}{2}\ln 2 - \frac{\pi}{4}i$

$\log(i) = \ln 1 + \frac{\pi}{2}i + 2\pi i n = \pi i\left(\frac{1}{2}+2n\right), n = 0, \pm1, \ldots$

(b) $\log i^{1/2} = \log e^{\frac{1}{2}\log i}$

$\log i = \pi i\left(\frac{1}{2}+2n\right)$ so $\log e^{\frac{1}{2}\pi i\left(\frac{1}{2}+2n\right)}$

$= \log e^{\frac{\pi i}{4} + n\pi i}$

$= \frac{\pi}{4}i + n\pi i + 2\pi m i \quad n, m$ integer.

$= \frac{\pi}{4}i + m\pi i \quad m = 0, \pm 1, \ldots$

Now $\frac{1}{2}\log i = \frac{1}{2}\left(\frac{i\pi}{2} + 2\pi n i\right) = i\frac{\pi}{4} + \pi n i$

and is the same.

Now $\log i^2 = \log -1 = \pi i + 2\pi n i$

$2\log i = 2\left[\frac{\pi}{2} i + 2\pi n i\right] = \pi i + 4\pi n i$

The first set contains $-\pi i$, but the second does not so $\log i^2$ and $2\log i$ are different sets.

3 (a) $(1+i)^i = e^{i\log(1+i)}$

$\log(1+i) = \ln\sqrt{2} + i\frac{\pi}{4} + 2\pi n i$

$(1+i)^i = e^{i\left[\frac{1}{2}\ln 2 + i\pi\left(\frac{1}{4}+2n\right)\right]} = e^{\frac{i}{2}\ln 2 - \pi\left(\frac{1}{4}+2n\right)}$

(b) $i^z = e^{z\log i} = e^{z\left[\frac{\pi}{2}i + 2\pi i n\right]}$

$= e^{\pi\left(\frac{1}{2}+2n\right)(ix-y)}$.

$= e^{-\pi\left(\frac{1}{2}+2n\right)y}\left(\cos\pi\left(\frac{1}{2}+2n\right)x + i\sin\pi\left(\frac{1}{2}+2n\right)x\right)$

$\mathrm{Re}(i^z) = e^{-\pi\left(\frac{1}{2}+2n\right)y}\cos\pi\left(\frac{1}{2}+2n\right)x$

$\mathrm{Im}(i^z) = e^{-\pi\left(\frac{1}{2}+2n\right)y}\sin\pi\left(\frac{1}{2}+2n\right)x$

where $n = 0, \pm 1, \pm 2, \ldots$

(4) $|\sin z|^2 = (\sin x \cosh y)^2 + (\cos x \sinh y)^2$

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So $|\sin z|^2 + |\cos z|^2 = (\sin^2 x + \cos^2 x)\cosh^2 y + (\sin^2 x + \cos^2 x)\sinh^2 y.$

$= \cosh^2 y + \sinh^2 y = 1 + 2\sinh^2 y$

Thus $|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y + \sin^2 x \sinh^2 y - \sin^2 x \sinh^2 y$

$= \sin^2 x (\cosh^2 y - \sinh^2 y) + (\cos^2 x + \sin^2 x)\sinh^2 y$

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Cleary ~~cosh~~ $|\sin z|^2 + |\cos z|^2 = 1 + 2\sinh^2 y \geq 1$
and $= 1$ ~~...~~ iff $\sinh y = 0$ $\quad e^y - e^{-y} = 0 \Rightarrow y = 0$
Sin. $e^y = e^{-y}$ if $y \neq 0$.

(5) $Re^{i\theta} = \tan z = \dfrac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}$

$= \dfrac{1}{|\cos z|^2} \sin z\, \overline{\cos z} = \dfrac{1}{|\cos z|^2} (\sin x \cosh y + i \cos x \sinh y)(\cos x \cosh y + i \sin x \sinh y)$

$= \dfrac{1}{|\cos z|^2} \left[(\sin x \cos x \cosh^2 y - \sin x \cos x \sinh^2 y) + i(\quad) \right]$

$\cos^2 x \sinh y \cosh y + \sin^2 x \cosh y \sinh y\ ]$

so $\tan\theta = \dfrac{(\sin^2 x + \cos^2 x)\sinh y \cosh y}{(\cosh^2 y - \sinh^2 y)\sin x \cos x} = \dfrac{\sinh y \cosh y}{\sin x \cos x} = \dfrac{\sinh 2y}{\sin 2x}$

Note: $\dfrac{(e^{y}-e^{-y})(e^{y}+e^{-y})}{4} = \dfrac{e^{2y}-e^{-2y}}{4} = \frac{1}{2}\sinh 2y.$

(6) $w = \tan^{-1} z \iff z = \dfrac{\sin w}{\cos w} = \dfrac{1}{i}\dfrac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}}$

or $e^{iw}-e^{-iw} = iz(e^{iw}+e^{-iw})$

Multiply by e^{iw}

$$(e^{iw})^2 - 1 = iz((e^{iw})^2+1)$$

$$e^{2iw} = \frac{1+iz}{1-iz}$$

$$w = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) = \tan^{-1} z$$

Thus $\tan^{-1} z_1 + \tan^{-1} z_2 = \frac{1}{2i}\left[\log\left(\frac{1+iz_1}{1-iz_1}\right) + \log\left(\frac{1+iz_2}{1-iz_2}\right)\right]$

Since $\log z_1 z_2 = \ln z_1 + \ln z_2$

$$\tan^{-1} z_1 + \tan^{-1} z_2 = \frac{1}{2i}\log\left(\frac{(1+iz_1)(1+iz_2)}{(1-iz_1)(1-iz_2)}\right)$$

$$= \frac{1}{2i}\log\frac{1+i(z_1+z_2)-z_1z_2}{1-i(z_1+z_2)-z_1z_2}$$

$$= \frac{1}{2i}\log\left(\frac{1+i\left(\frac{z_1+z_2}{1-z_1z_2}\right)}{1-i\left(\frac{z_1+z_2}{1-z_1z_2}\right)}\right) = \tan^{-1}\left(\frac{z_1+z_2}{1-z_1z_2}\right)$$