**Intro. to Math. Modeling ANSWERS TO REVIEW PROBLEMS (Typos corrected)May 2, 2005**

1. (a) The birth rate is $250/1000 = .25$ and the death rate is $220/1000 = .22$. Thus $R = R_0 = .25 - .22 = .03$. Thus $N(t + \Delta t) = (1 + R_0)N(t) = 1.03N(t)$.

(b) After 10 years, starting from 5000, the populations will be $(1.03)^{10}5000 = 6720$.

(c) $N(t + \Delta t) = 1.03N(t) + .5$, since $N$ is measured in 1000's.

(d) Recall $N_{k+1} = rN_k + m$, may be solved by assuming $N_k = Ar^k + B$, from which we have $B = m/(1 - r)$ and $A = N_0 - m/(1 - r)$. In the present problem $r = 1.03, m = .5$, and $N_0 = 5$. Thus $N_k = (5 + .5/.03)(1.03)^k - .5/.03$. With $k = 10$ we obtain 12452 individuals.

2. (a) In $T$ years where $2 = e^{.03T}$, or $T = 23.1$ years.

(b)$\frac{dN}{dt} = .03N + .5t$, $N(0) = 5$. Inserting $N = Ae^{.03t} + Bt + C$, we see that $B = .03(Bt + C) + .5t$. Thus $.03B + .5 = 0$ and $B = .03C$ so $B = -16.666$ and $C = -555.555$. Then to make $N(0) = 5$ we must have $A = 560.555$. After 10 years the population is therefore $560.555e^{.3} - 166.66 - 555.555 = 34.45$ or 34500 individuals approximately.

3. (a) $\frac{dN}{1+N} = tdt$, and this integrates to give $ln(1 + N) = t^2/2+$ constant or $N = Ce^{t^2/2} - 1$ From the initial condition $C - 1 = 1$ or $C = 2$.

(b)The integrating factor is $e^{\int tdt} = e^{t^2/2}$ and so $\frac{d}{dt}e^{t^2/2}x = te^{t^2/2}$. This integrates to $x = 1 + Ce^{-t^2/2}$, and the initial condition implies $C = 1$.

(c) The equation integrates once to give $\frac{dx}{dt} + x = C$. From the two initial conditions $C = 3$. Clearly $x = 3$ is a particular solution of the resulting equation. Thus $x = 3 + Ae^{-t}$. The condition $\frac{dx}{dt}(0) = 2$ implies $A = -2$.

4. (a) $r$ is a fraction of space, It has no units. The units of $F$ are $feet/sec \times feet^2 = feet^3/sec$.

(b)If $A$ is one square foot and the level is observed to rise 1/12 foot in one hour = 3600 seconds, then $F$ must be $\frac{1}{12\times 3600}$ cubic feet per second. Then $r = \frac{F}{UA} = (12 \times 3600 \times 15 \times 1)^{-1} = .1543 \times 10^{-5}$.

(c) The rainfall is steady so you will be running through space with water occupying a fraction $r$ of the space. The velocity of 5 feet per second and the frontal area of 10 sq ft means that the flux onto your front will be $10 \times 5 \times r = 50r = 7.715 \times 10^{-5}$ cubic feet per second. There are $12^3 = 1728$ cubic inches in a cubic foot, and 60 seconds per minute. Multiplying by these two numbers converts cubic feet per second to cubic inches per minute. Thus you will pick up about 8 cubic inches of water per minute.

5. (a) The possible equilibria are 0, 1000, and 3000 individuals. The equilibrium at $N = -2$ is not feasible since population size cannot be negative.

(b) If the equation is $\frac{dN}{dt} = F(N)$, and equilibrium $N_e$ is stable if $dF/dN(N_e < 0$, and is otherwise unstable. We compute $dF/dN(0) = -6, dF/dN(1) = 6, dF/dN(3) = -30$. Thus 1 is unstable and the other two are stable.

6. (a) The equilibrium populations are 0 and $(1 - 1/r)^{1/3} \times 1000$ individuals.

(b) For a recursion of the form $N_{k+1} = F(N_k)$, an equilibrium is stable if $|dF/dN(N_e)| < 1$. Otherwise it is unstable. In the present case $dF/dN(0) = r > 1$ so 0 is always unstable. Also $dF/dN((1 - 1/r)^{1/3}) = r(1 - 4N_e^3) = 4 - 3r$. This equilibrium is thus stable for $1 < r < 5/3$, and is unstable for $r > 5/3$.

7. (a) In the notation of the text, $b_0 = 1, b_1 = 1/3, d_0 = 1/4$, so the matrix is $A \begin{pmatrix} 1 & 1/3 \\ 3/4 & 0 \end{pmatrix}$.

(b) $det(A - \lambda I) = -\lambda(1 - \lambda) - 1/4$, so the eigenvalues are $\lambda = \frac{1}{2}(1 \pm \sqrt{2})$ The population will grow by the factor $(1 + \sqrt{2})/2 = 1.207$. The corresponding eigenvector satisfies

$$\begin{pmatrix} \frac{1-\sqrt{2}}{2} & 1/3 \\ 3/4 & -\frac{1+\sqrt{2}}{3} \end{pmatrix} \cdot \mathbf{N} = 0.$$

Thus $N_2/N_1 = \frac{3}{2}(\sqrt{2} - 1) = .62$ gives the ultimate distribution between days 1 and 2.

8. (a) $N_1$ growth reduces its own growth rate in a logistic fashion, and also the growth rate of $N_2$. $N_2$ enhances the growth of $N_1$ but makes a logistic reduction of its own growth. It would be fair to charaterize $N_2$ as prey for $N_1$, since $N_1$ is bad for $N_2$ but $N_2$ is good for $N_1$.

(b) (In the statement of the problem $x = N_1, y = N_2$. On the line $N_1 = 0$ we have $\frac{dN_1}{dt} = 0$. Similarly on $N_2 = 0$ we have $\frac{dN_2}{dt} = 0$, so the integral curves cannot penetrate either of these sides. On $N_1 = 3, 0 \le N_2 \le 2$, we see that $\frac{dN_1}{dt} \le 0$. Also on $N_2 = 2, 0 \le N_1 \le 3$ we see that $\frac{dN_2}{dt} \le 0$. Therefore the solution curves cannot be leaving $R$ on these sides. Thus any solution curve starting from a point in $R$ must stay in $R$.

(c) The non-zero equilibrium is $(3/2, 1/2)$. Perturbations around this equilibrium satisfy $\frac{d\delta\mathbf{N}}{dt} = A \cdot \delta\mathbf{N}$ where $A = \frac{\partial F_i}{\partial N_j}(\mathbf{N}_e)$. This works out to

$$A = \begin{pmatrix} -3/2 & 3/2 \\ -1/2 & -1/2 \end{pmatrix}.$$

We compute $Det(A - \lambda I) = \lambda^2 + 2\lambda + 3/2 = 0$. The roots are $\lambda = -1 \pm i/\sqrt{2}$. This equilibrium is stable and the imaginary part of the roots indicates that near the equilibrium the solution curves are spiralling into $(3/2, 1/2)$.

(d) We expect to find that every integral curve goes to $(3/2, 1/2)$, indicating this is a globally stable equilibrium of the two species.

9. (a) If $0 < x_k, y_k < 1$ we see that $x_{k+1} < 1/3$ and then $y_{k+1} < 1$. So the minimums in the bb-equation yield the stated equation.

(b)$y_{k+2} = x_{k+1} + \frac{2}{3}(1 - y_{k+1}) = \frac{1}{3}(1 - y_k) + \frac{2}{3}(1 - y_{k+1})$, or $y_{k+2} + \frac{2}{3}y_{k+1} + \frac{1}{3}y_k = 1$. The equilibrium is a constant solution of this equation, which is $y_e = 2$. To find solutions of $y_{k+2} + \frac{2}{3}y_{k+1} + \frac{1}{3}y_k = 0$, set $y_k = r^k$. The $r$ must satisfy $r^2 + \frac{2}{3}r + \frac{1}{3} = 0$ or $r = r_{1,2} = -\frac{1}{3} \pm i\frac{\sqrt{2}}{3}$. For either $r$ we see that $|r| = 1/\sqrt{3} < 1$ and so the general solution representing the system, of the form $y_k = y_e + A_1 r_1^k + A_2 r_2^k$, converges to $y_e$ for large $k$. The coresponding equilibrium value of $x$ is $x_e = 1/6$, since $x_e = \frac{1}{3}(1 - y_e)$.

10. From $dx/dt = x^2/(1+t)$, solution by separation of variables gives $1/x + \ln(1+t) = \phi$=constant. The solution is of the form $f = F(\phi)$. We want $f(x, 0) = x$, but when $t = 0$, $\phi = 1/x$. Therefore $F(\phi) = 1/\phi$ and $f = \frac{x}{1 + x\ln(1+t)}$.

11. Here $v(\rho) = 60(1 - \rho/150)$. Thus in $x < 0$, where initially $\rho = 200$, $v(\rho) = 60(1 - 4/3) = -20mph$. The charateristics emerging from point $x_0$ in $x < 0$ therefore have the equation $x = -20t + x_0$. Similarly in $x > 0$, where $\rho = 50$, we have $v(\rho) = 60(1 - 1/3) = 40mph$ and so the equation of the characteristics is $x = 40t + x_0$.

(b)We want to solve $\frac{\partial\rho}{\partial t} + 60(1 - \frac{\rho}{150})\frac{\partial\rho}{\partial x} = 0$. With $\rho = R(x/t)$ we have $-R'\left[-x/t^2 + 60(1 - R/150)/t\right] = 0$ giving $\rho = R = \frac{5}{2}(60 - x/t)$. This gives (200,50) when $x/t$= (-20,40), as required.

12. The substituion gives $m\lambda^2 + \mu\lambda + k = 0$, so $\lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m}$. Since $mk > 0$ we see that when $\mu^2 - 4mk > 0$ we have two negative real roots $\lambda_1, \lambda_2$. The general solution is then $x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$ and both terms decay to zero as simple exponentials.

When $\mu^2 < 4mk$ the roots are $\frac{-\mu}{2m} \pm i\omega$ where $\omega = \frac{1}{2m}\sqrt{4km - \mu^2}$. The general solution is now $A_1 e^{-\mu/2m}\cos\omega t + A_2 e^{-\mu/2m}\sin\omega t$, which corresponds to a decaying oscillation.