Applied math II Spring 2003

The bursting balloon reconsidered

Kirchoff's solution of the IVP for the 3D wave equation is

$$u(\mathbf{x},t) = \frac{1}{4\pi c^2 t}\int\int_{S(\mathbf{x},t)} u_t(\mathbf{x},t)dS' + \frac{\partial}{\partial t}\Big[\frac{1}{4\pi c^2 t}\int\int u(\mathbf{x}',0)dS'\Big].$$

Here S is the spherical survace with center at $\mathbf{x}$ and radius ct and dS' indicates that the point $\mathbf{x}'$ is integrated over the surface S.

We now reconsider the bursting balloon and recover the solution using the Kirchoff formula. Here u is the pressure p. The firgure below indicates the geometry. The law of cosines implies

$$r_b^2 = R^2 + (ct)^2 - 2Rct\cos\alpha.$$

The surface of intersection of S with the balloon has area $2\pi(1-\cos\alpha(ct)^2$. This is easily established in spherical coordinates. (This is a calculus exercise worth doing if you don't recall this.) Now the only term we need to consider in the Kirchoff formula is the second one. Since

$$1-\cos\alpha = \frac{r_b^2-(R-ct)^2}{2Rct},$$

we obtain

$$u = p = \frac{\partial}{\partial t}\Big[\frac{1}{4\pi c^2 t}p_b(2\pi c^2 t^2)\frac{r_b^2-(R-ct)^2}{2Rct}\Big].$$

$$p = \frac{\partial}{\partial t}\frac{1}{4c}\frac{p_b}{R}(r_b^2-(R-ct)^2)$$

$$= \frac{1}{2}\frac{R-ct}{R}p_b.$$

Here $R + r_b > ct > R - r_b$. For any other value of R the area of intersection with the balloon is *zero* and so $p = 0$.This is the result we obtained previously by reducing the balloon problem to an IBVP for the one-dimensional wave equation in spherical coordinates.