Notes on the bucket-brigade production line

We consider a production line which produces widgets. The line consists of a series of machines where operations are performed. We take the line to have unit length, and the position of the kth worker on the line to be x_k. We assume each worker moves along the line with a *constant* velocity v_k. Then $1/v_k = T_k$ is the time it would take worker k alone to produce one widget.

In addition to constant velocity, the following assumptions are made: Workers do not "cross over", i.e. pass each other, on the line. Imagine the workers move from left to right. Then a fast worker who meets up with a slower worker will be *blocked* by the slower worker. That is, the faster worker will move along the line at the velocity of the slower worker.

Suppose there are n workers, worker 1 on the left up to worker n on the right. We say a *reset* occurs when the nth worker finishes the widget. At a reset, worker n takes over the widget held by worker $n-1$, worker $n-1$ takes the widget from $n-1$, etc., worker 2 from worker 1, and then worker 1 introduces a new widget (or whatever starts the process) into the system. We assume the reset occurs *instantaneously*. Of course, in fact some time is needed to reset, so the assumption is that this actual reset time is small compared to the manufacturing time for the widget.

The system can be thought of a starting with workers in fixed positions on the line, waiting for a widget. Worker 1 starts the process, and when worker 2 is encountered hands off the item and takes a new one, while worker 2 continues until encountering worker 3, hands off, takes from 2, 2 from 1, etc. Eventually all workers have widgets, and this is where we begin to observe the dynamics of the production line.

We begin our observation at a reset. In this case we know that $x_1 = 0$, so the *state* of the line at reset is determined by the $n-1$-vector $X = [x_2, x_3, \ldots, x_n]$, with $0 \leq x_2 \leq x_3 \leq \ldots \leq_n \leq 1$. We can thus describe the performance of the line by the discrete sequence of vectors $X^{(k)}$ giving worker positions at reset times, and by the sequence of reset times $T^{(k)}$. For example the average production rate of widgets would be

$$P \equiv \lim_{M\to\infty} \frac{M}{T^{(1)} + T^{(2)} + \ldots + T^{(M)}}.$$

An example: Consider a two-person bucket-brigade production line. Worker 1 can produce a widget in 1 hour, worker 2 in 1/2 hour. The $v_1 = 1, v_2 = 2$. (Note: the actual units of time are not important here. It will be clear below that only the *relative velocities* of workers is important, here that worker 2 is twice as fast as worker 1.) At the reset time worker 1 is at $x_1 = 0$ and worker 2 is at $x_2^{(1)}$. We may assume $0 < x_2^{(1)} < 1$.

In the figure below, we show an $x - t$ diagram of the subsequent worker positions, assuming $x_2^{(1)} = 1/2$

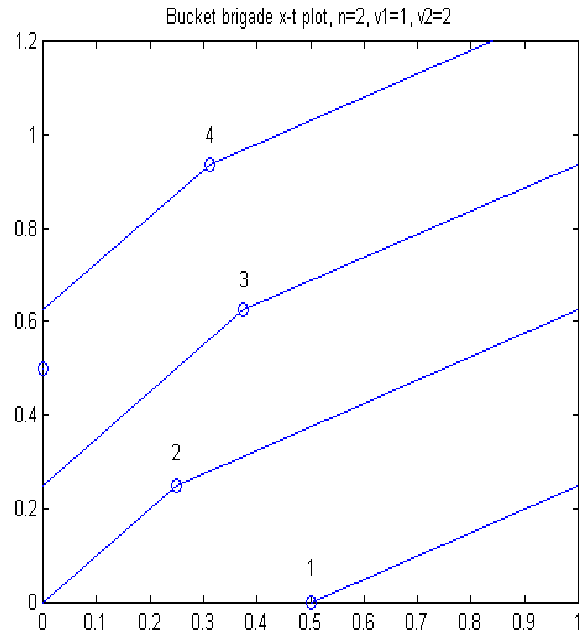

Figure 1. Two-person bucket brigade with $v_1 = 1, v_2 = 2$, and $x_2^{(1)} = .5$.

Here the points marked 1-4 are $(x_2^{(k)}, T^{(k)}), k = 1, 2, 3, 4$ and give the times and locations at reset. There are respectively $1/2, 0), (1/4, 1/4), (3/8, 5/8), (5/16, 15/16)$.

We can easily compute these values of $x_2^{(k)}$ from the resets. We see that

$$x_2^{(k+1)} = \frac{v_1}{v_2}(1 - x_2^{(k)}) = \frac{1}{2}(1 - x_2^{(k)}).$$

We thus get the first order inhomogeneous recursion $2x_2^{(k+1)} + x_2^{(k)} = 1$. The solution is $x_2^{(k)} = 1/3(1 - (-1/2)^k)$. We thus see that

$$\lim_{k \to \infty} = 1/3.$$

with $x_2 = 1/3$ at reset wee see that the production line repeats itself with every reset. We say that the line is *balanced* when the equilibrium is reached. In this balance case a widget is finished every 1/3 of a time unit, or we get 3 widgets per time unit as the production rate. We can see from the recursion that $x_2^{(k)} \to \frac{v_1}{v_1 + v_2} \equiv x_2^*$ (verify this!), which means that a widget is produced every $x^*/v_1 = \frac{1}{v_1 + v_2}$ time units, or $v_1 + v_2 = 3$ widgets are produced every time unit.

The main theorem of this model of the BB(Bartholdi *et al.* 1995): *If an n-worker BB is sequenced so that $v_1 < v_2 < v_3 < \ldots < v_n$, then the line is balanced, and the production rate at equilibrium is $P = v_1 + v_2 + \ldots + v_n$ items per unit time. This production rate is optimal over all possible sequencing of workers.* We omit proof of this and continue with examples. We remark that the sequencing $v_1 < v_2 < v_3 < \ldots < v_n$ is not a necessary condition for optimal production.

Consider our example with the sequence of two workers reversed, $v_1 = 2, v_2 = 1$. With again $x_2^{(1)} = 1/2$, we get the $x - t$ diagram shown below.

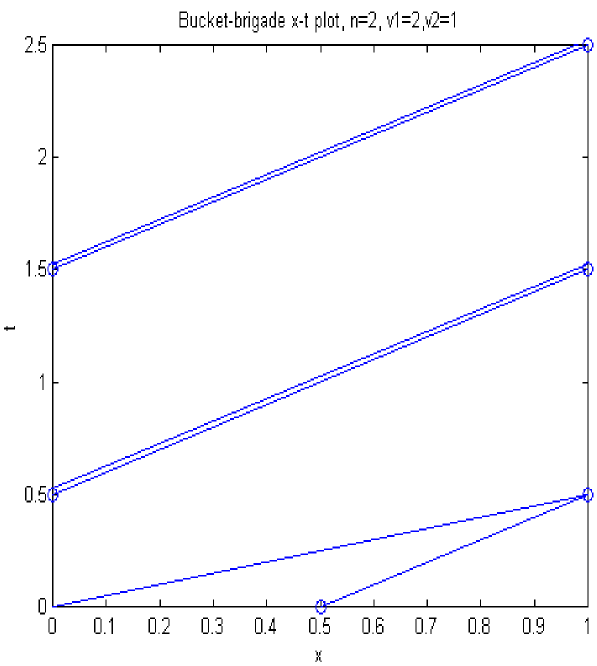

Figure 2. Two-person bucket brigade with $v_1 = 2, v_2 = 1$, and $x_2^{(1)} = .5$.

Now we see that the first reset occurs at $t = .5$ when both workers arrive at $x = 1$. After the reset 2 is holding 1's widget at $x = 1$ and 1 is back at $x = 0$ with a new widget. Thus $x_2^{(2)} = 1$. Immediately 2 finishes the widget and rests by taking 1's widget at $x = 0$. Thus both are holding widgets at $x = 0$ after the reset and so $x_2^{(3)} = 0$. Now 1 is blocked by the slower 2 and both proceed at velocity $v_2 = 1$ along the line until the next reset at $t = 1.5$. The fact that two resets occur instantly, followed by another reset time unit 1 later, means that two completed widgets are delivered (simultaneously) per unit time, so $P = 1$ and the production rate is suboptimal. (Note that $P = 2v_2$ here, compared with the optimal value of $v_1 + v_2$.)

For a two-worker BB the two cases we have examined may be summarized in a difference equation for $x_2^{(k)}$ which contains both. Note that if blocking occurs a time t_b following a reset (where 2 was at $x_2^{(k)}$ say, and necessarily with $v_1 > v_2$), at x_b say, this means that $x_b = v_1 t_b = x_2^{(k)} + v_2 t_b$. Thus $t_b = \frac{x_2^{(k)}}{v_1 - v_2}$ and $x_b = \frac{v_1 x_2^{(k)}}{v_1 - v_2}$. Since $x_b \leq 1$ we have $v_1 x_2^{(k)} \leq v_1 - v_2$ and this implies $\frac{v_1}{v_2}(1 - x_2^{(k)}) \geq 1$. Since blocking at a point $x_b > 0$ leads to $x_2^{(k+1)} = 1$ we can replace the recursion by the general relation $x_2^{(k+1)} = \min[\frac{v_1}{v_2}(1 - x_2^{(k)}), 1]$.

Note that once the "1" is obtained in this minimum, the sequence $1, 0, 1, 0, 1, \ldots$ begins.

The three-or-more-worker BB: We consider some aspects of the three-worker BB. After reset k, each worker have time $(1 - x_3^{(k)})/v_3$ before the next reset. During that time the first two workers cannot proceed further than the distance they can travel in time $(1 - x_3^{(k)})/v_3$ at their respective velocities, or else by the final positions of the workers to their right, if they are blocked. We thus can obtain the dynamics from a chain of minima:

$$x_3^{(k+1)} = \min[x_2^{(k)} + \frac{v_2}{v_3}(1 - x_3^{(k)}), 1]$$

,

$$x_2^{(k+1)} = \min[\frac{v_1}{v_3}(1 - x_3^{(k)}), x_3^{(k+1)}].$$

It should be clear that we can extend this argument to the n-worker line. Problem 4 of homework 7 will explore an example of this, which shows that optimal production does not require monotone ordering from slowest to fastest workers.

For $n \geq 4$ the line's dynamics can be chaotic. We show an example in the next figure.

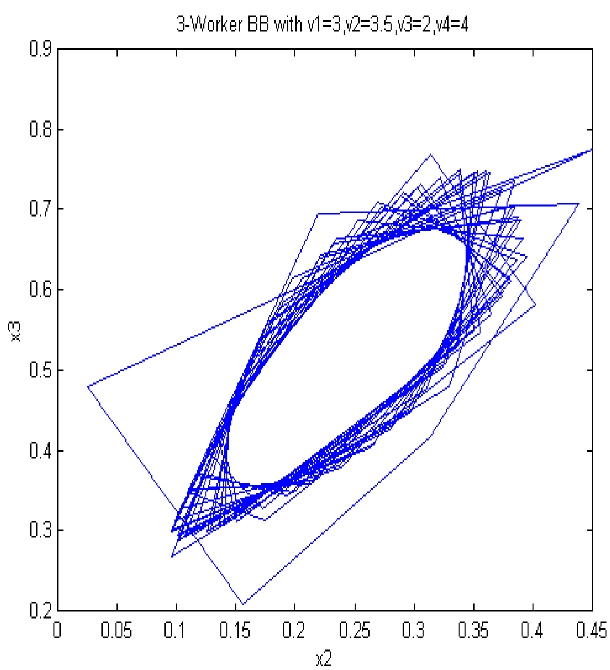

Figure 2. Chaotic behavior in a 4-person bucket brigade with $v_1 = 3, v_2 = 3.5, v_3 = 2, v_4 = 4$. The plots x_2 versus x_3.