1

Introduction

1.1 General considerations

In this monograph we shall deal with the movement of things through fluids, and with the fluid domain as a medium of transport and locomotion. These topics comprise a vast but basic branch of fluid mechanics. They include swimming and flying in fluids, fluid pumping, turbulent transport, and mixing. Our aim is to provide a unified treatment of a subset of problems of this type, and our first task is to indicate a little more precisely the focus of our study. The material will deal largely with mathematical theory and models, but will also treat various applications to practical problems. We also view the topics as a kind of self-contained course in the fundamental, if somewhat advanced fluid mechanics needed to cope with devices which move or move in a fluid.

The first observation we can make is that transport or locomotion in a fluid medium is quite different from walking, driving a car, or hauling a freight train, all of which enables both transport and locomotion in our daily lives. The free swimmer, for example, advances through the water by "pushing" on it, thus causing the water to move as well, so that the locomotion has to be viewed as an interaction of body and fluid, which finally results in net displacements of both. The transport of material along a conveyor belt or by a bucket brigade also seems fundamentally different from what can be achieved in a fluid medium. How can this kind of direct transport emerge in a continuously varying fluid? Indeed, is this kind of direct control on transport even possible in a fluid? And what characterizes the fluid motions which can move suspended material in the absence of any diffusional processes. Diffusion-independent transport is observed in fluid turbulence, where the fluid motion is highly chaotic. Are there nevertheless properties of the flow directly linked to transport?

If we ask how to distinguish between locomotion and transport, or indeed how these terms should be defined, we are lead rather directly to the realization that to some extent they are two realizations of basically the same physical processes. We will try to develop this idea, restricting the discussion to two space dimensions, with the help of Figure 1.1. In 1.1(a) we suggest a definition of “transport” as the bulk motion of a fluid contained within a domain whose boundary moves, the boundary itself not undergoing any net displacement. We might call this the “stirring motion” of the boundary. In 1.1(b) this is made into a better representation of a pump by allowing the domain to be multiply connected, the boundary motion now causing a net flow through the annular region between the stationary and moving wall. The stirring motion might be such that each point on the moving boundary executes (for example), a closed periodic motion, without any net displacement over one cycle.

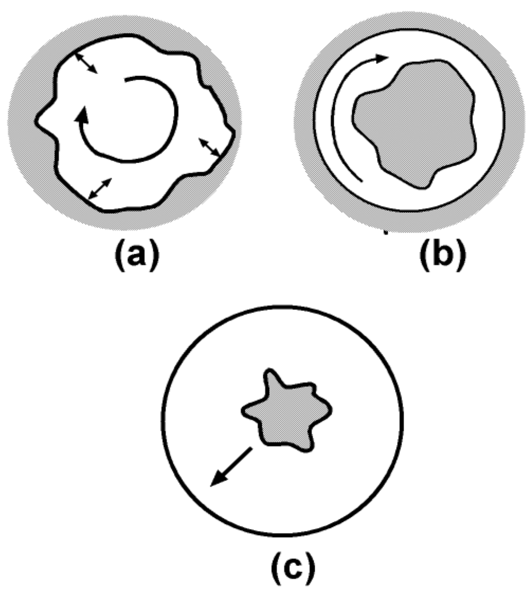

Figure 1.1: In (a) the boundary movement causes net motion of the fluid. (b) A variant of (a) in a multiply-connected domain. (c) Locomotion of a body in a bounded fluid domain.

In Figure 1.1(c) we illustrate locomotion, the inner, moving boundary being now free to move about. Again the points of the body can be thought of a executing closed periodic orbits when the body is"held fixed", but in its free state there is presumably net displacement and we speak of the body as flying or swimming. [1] If we compare (a) and (c) we see that in this example "transport" and "locomotion" differ in two ways. First, the two configurations are essentially geometrical inversions, the fluid being inside of the boundary on the one hand and outside of it on the other. Secondly, and more important to the distinction, on the one hand the moving boundary is "fixed", and in the other it is "free". By relaxing this distinction we could then consider a single class of flows involving points associated with "boundary" or "body" and points associated with "fluid". By conditioning the former to be fixed or free, both transport and locomotion can be modeling in a single framework, by focusing on the bulk motion of one or the other set of points.

This description suggests that these problems should be approached as continuum mechanics of an inhomogeneous medium consisting of both fluid and solid components, a viewpoint that has in fact proved to be valuable in both theoretical modeling and numerical simulation. A natural swimmer such as a fish might be modeled in a way that accounts for the body's mass, muscle structure, flexibility, and shape. Supplied with a source of energy sufficient to maintain its movement, the swimming must be understood as a result of the dynamical interaction of fluid and body. This unified, but somewhat extreme view of our class of problems can be accepted if it is understood that in many situations the "non-fluid" material can be adequately dealt with in a simpler manner, for example by supplying a given boundary motion whenever the reaction of the fluid is not to be considered. In some cases the fluid motion itself can be supplied, for example when studying transport of a passive scalar in a given velocity field.

The view of locomotion and transport as two aspects of the same problem will permeate the organization of this monograph, even though we will treat the topics in two distinct Parts. A further basic division will be made, on the basis of the relative importance of the viscosity of the fluid, into the *Stokesian realm* and the *Eulerian realm*. In the Stokesian realm viscous forces dominate inertial forces, the boundary and fluid are intimately linked by the rapid diffusion of momentum, and similar mechanisms apply whether the boundary locomotes or drives a flow. In the Eulerian realm viscous stresses are nominally negligible, but in fact the role of viscosity is extremely subtle. This is particularly true in the case of locomotion, where effective mechanisms involve the creation of vorticity in the fluid, fundamentally a viscous process in the problems we study here.

[1] We shall return below to the distinction between flying and swimming. In brief, the weight of the object is of no consequence in swimming, the density of the water allowing for buoyant lift.

1.2 Kinematics

With the present section we begin our introduction to some basic methods of fluid dynamics. We shall make an assumption which will apply globally to this work. In any expression involving functions and their derivatives, the necessary smoothness is assumed. However, for any calculation where regularity issues are important in the physicial problem under study, the necessary assumptions on functions wiull be made explicit.

1.2.1 Lagrangian coordinates

Some considerations of importance to us will not depend upon the dynamical processes responsible for the motion of fluid or body. In this case we simply study the motion of a point associated with body or fluid, without regard to the causes of the motion. The *kinematics* of motion of a given point may be expressed as a vector function $\mathbf{x}$ of time t and a variable $\mathbf{a}$ which serves to identify the chosen point. In all of our work $\mathbf{x}$ will lie in two or three space dimensions (variously (x, y, z) or x_1, x_2, x_3), and $\mathbf{a}$ will have the same number of dimensions as the object under study. Often the label $\mathbf{a}$ is taken as the initial position of the point relative to some standard reference frame, and we refer to this case as *standard* Lagrangian coordinates. We use $\mathcal{R}^N$ to denote Euclidean space in N dimensions.

We shall always assume that $\mathbf{x}$ is a continuous function of $\mathbf{a}, t$. If it is also a differentiable function of time, then we may define the *velocity* of the point at time t as

$$\mathbf{u} \equiv \frac{\partial \mathbf{x}}{\partial t}\Big|_{\mathbf{a}} \equiv \frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{a}, t). \tag{1.1}$$

We refere to $\mathbf{u}(\mathbf{a}, t)$ as the Lagrangian velocity field.

If the chosen point is one which is transported relative to the reference frame, or else locomotes through the fluid relative to that frame, then the values of $\mathbf{x}$ as time increases will lie in some non-trivial point set, and also $\mathbf{u}$ will have non-zero values. In the case of locomotion it will often be convenient to define the reference frame as"attached to the body", i.e. it is a body frame, so that body points execute limited excursions relative to the frame. In this case the locomotion of the body will be determined by the kinematics of the body frame itself, relative to a standard reference frame, usually one that is fixed relative to distant fluid.

Such a function $\mathbf{x}(\mathbf{a}, t)$ is said to be the *Lagrangian coordinate* of the point determined by the given value of $\mathbf{a}$. It is important that we will consider open sets over which $\mathbf{a}$ is defined, for this allows us to speak of a deformable body or a blob (parcel) of fluid , and hence of the locomotion and transport of finite amounts of material. When dealing with Lagrangian coordinates we let $\mathcal{D}$ denote the domain of values of the variable $\mathbf{a}$. If $\mathbf{a}$ is defined customarily as an initial value, then $\mathcal{D}$ is the initial configuration of Lagrangian points whose motion is determined by $\mathbf{x}$. We may then enlarge the discussion to include the evolution of the configuration with time. We will often use the term *material* as equivalent to Langrangian in reference to coordinates. Material point, surfaces, or fluid parcels are thus collections of Lagrangian points, moving with the fluid.

Indeed we may under the above assumptions on $\mathbf{x}$ regard Lagrangian coordinates as a map

$$\mathcal{M} : \mathcal{D} \to \mathcal{R}^N. \tag{1.2}$$

We shall assume that $\mathbf{x}$ is a differentiable function of $\mathbf{a}$ for $t > 0$, so that we may form the Jacobian tensor $\mathbf{J}$:

$$\mathbf{J}(\mathbf{a}, t) = (J_{ij}) = \Big(\frac{\partial x_i}{\partial a_i}\Big). \tag{1.3}$$

The Jacobian tensor, when multiplied on the left by the differential column vector $d\mathbf{a}$, yields the image of $d\mathbf{a}$ under the flow, allowing us to follow the rotation and elongation of the separation vector connecting two nearby points. Also the determinant $|\mathbf{J}|$ of the Jacobian determines the effect of $\mathcal{M}$ on small volume elements. In particular for the important case of a volume preserving map $\mathcal{M}$, we have $|\mathbf{J}| = 1$.

1.2.2 Eulerian fields

Although we have used the term already to indicate a realm of fluid dynamics where viscous stresses are nominally small, we use *Eulerian* here to indicate an alternative formulation of continuum mechanics in terms of fields which are given as functions of $\mathbf{x}, t$ defined on $\mathcal{R}^N \times \mathcal{R}$. We then may refer to $\mathbf{x}$ as *Eulerian coordinates.* We again use $\mathcal{D}$ to denote the domain, now for the Eulerian coordinate $\mathbf{x}$. In such a scheme the tracking of the position of individual points of $\mathcal{D}$, the basis of the Lagrangian picture, is replaced by a "snapshot" of the fields– velocity, temperature, etc.– at each instant of time over the entire domain $\mathcal{D}$. We shall always assume these fields are sufficiently smooth to allow the operations of integration, and more importantly, of differentiation, that we carry out below. The Eulerian velocity field $\mathbf{u}_e(\mathbf{x}, t)$ is thus related to the set of Lagrangian coordinates $\mathbf{x}(\mathbf{a}, t)$ by the following equation:

$$\frac{\partial \mathbf{x}}{\partial t}\Big|_{\mathbf{a}} = \mathbf{u}_e(\mathbf{x}(\mathbf{a}, t), t) = \mathbf{u}(\mathbf{a}, t). \tag{1.4}$$

If $\mathbf{u}_e(\mathbf{x}, t)$ is known, (1.4) is a system of N ordinary differential equations, whose solution space defines the totality of Lagrangian coordinates for the given flow.

In general differentiation of an Eulerian field $Q(\mathbf{x}, t)$ with $\mathbf{a}$ held fixed yields

$$\frac{dQ}{dt}\Big|_{\mathbf{a}} = \frac{\partial Q}{\partial t}\Big|_{\mathbf{x}} + \frac{\partial x_i}{\partial t}\Big|_{\mathbf{a}} \frac{\partial Q}{\partial x_i} = Q_t|_{\mathbf{x}} + \mathbf{u}_e \cdot \nabla Q \equiv \frac{dQ}{dt}. \tag{1.5}$$

We thus speak of dQ/dt as the derivative following a Lagrangian point, of the *material derivative* with respect to time. For example, the Eulerian form of the *fluid acceleration* is defined by

$$\frac{d\mathbf{u}}{dt} \equiv \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}. \tag{1.6}$$

If we differentiate (1.4) with respect to $\mathbf{a}$, there results, using the chain rule in component form,

$$\left.\frac{dJ_{ij}}{dt}\right|_{\mathbf{a}} = \frac{\partial u_i}{\partial x_k} J_{kj}. \tag{1.7}$$

Using (1.5) we obtain the Eulerian equation for the Jacobian tensor as a function of $\mathbf{x}, t$:

$$\mathbf{J}_t + \mathbf{u} \cdot \nabla \mathbf{J} = \mathbf{D} \cdot \mathbf{J}, \mathbf{D} = [\partial D_i/\partial x_j]. \tag{1.8}$$

Here and often below we shall omit the subscript e when $\mathbf{x}$ is understood to be the independent variable. The corresponding equation for the determinant $|\mathbf{J}|$ is (see Exercise 1.1)

$$|\mathbf{J}|_t + \mathbf{u} \cdot \nabla |\mathbf{J}| = (\nabla \cdot \mathbf{u})|\mathbf{J}|. \tag{1.9}$$

It follows that a volume-preserving Lagrangian map $\mathcal{M}$ determines an Eulerian velocity field satisfying $\nabla \cdot \mathbf{u} = 0$. We say that such a flow is *incompressible*. Since air at speeds well below that of sound, as well as water, are essentially incompressible fluids, our discussion will be restricted for the most part to divergence-free velocity fields $\mathbf{u}$.

1.3 Conservation laws

To study conserved quantities in fluid flows, we need to consider properties associated with fluid parcels, which by definition are associated with Lagrangian points, moving with the fluid. Let $\mathcal{D}_t$, be such a fluid parcel, bounded by an orientable material surface $\partial\mathcal{D}_t$, and moving freely in $\mathcal{R}^N$ with the Eulerian velocity field $\mathbf{u}$, and let Q be a property of the fluid defined at each point of $\mathcal{D}_t$. We then consider the time derivative of the integral of Q over $\mathcal{D}_t$:

$$\frac{d}{dt}\int_{\mathcal{D}_t} Q(\mathbf{x}, t) dV(\mathbf{x}). \tag{1.10}$$

We here write $dV(\mathbf{x})$ to indicate the independent variables of the volume integration. Since $dV(\mathbf{x}) = |\mathbf{J}|dV(\mathbf{a})$, it is useful to employ Lagrangian coordinates and write so as to obtain

$$\frac{d}{dt}\int_{\mathcal{D}_t} Q(\mathbf{x}, t) dV(\mathbf{x}) = \frac{d}{dt}\int_{\mathcal{D}_0} Q(\mathbf{x}, t)|J| dV(\mathbf{a})$$

$$= \int_{\mathcal{D}_0} \big[dQ/dt|\mathbf{J}| + Q\frac{d|\mathbf{J}|}{dt}\big]dV(\mathbf{a}). \tag{1.11}$$

Note that the material derivative of Q appears here inside the integral, since Q is a being regarded as a function of $\mathbf{a}, t$ following the change of variable to $\mathbf{a}$. Using (1.9) we then have

$$\frac{d}{dt}\int_{\mathcal{D}_t} QdV(\mathbf{x}) = \int_{\mathcal{D}_t} [dQ/dt(\mathbf{x},t) + Q\nabla\cdot\mathbf{u}]dV(\mathbf{x}). \tag{1.12}$$

If Q in $\mathcal{D}$ is *conserved*, the

$$\frac{d}{dt}\int_{\mathcal{D}_t} QdV(\mathbf{x}) \equiv 0. \tag{1.13}$$

This then implies, assuming that the integrand is a continuous function of $\mathbf{x}$, that

$$dQ/dt(\mathbf{x},t) + Q\nabla\cdot\mathbf{u} = \frac{\partial Q}{\partial t} + \nabla\cdot(\mathbf{u}Q) = 0. \tag{1.14}$$

Here we have combined terms using $\mathbf{u}\cdot\nabla Q + Q\nabla\cdot\mathbf{u} = \nabla\cdot(Q\mathbf{u})$. The form (1.14) is referred to as the *local form* of conservation of Q. Note that, if the divergence theorem wass used we could have written

$$\frac{d}{dt}\int_{\mathcal{D}_t} QdV(\mathbf{x}) = \int_{\mathcal{D}_t} \frac{\partial Q}{\partial t}(\mathbf{x},t)dV(\mathbf{x}) + \int_{\partial\mathcal{D}_t} Qu_n dS(\mathbf{x}), \tag{1.15}$$

where u_n is the outward normal component of $\mathbf{u}$ evaluated on $\partial\mathcal{D}_t$. We see that (1.15) expresses conservation of Q as a balance between the time variation of Q over $\mathcal{D}_t$ and the flux of Q through the boundary $\partial\mathcal{D}_t$.

The above Lagrangian derivation of (1.14) should be contrasted with the corresponding Eulerian argument. Here we consider a *fixed* but arbitary domain $\mathcal{D}$ and and explicitly balance the time derivative with a flux term:

$$\frac{d}{dt}\int_{\mathcal{D}} QdV(\mathbf{x}) = \int \frac{\partial Q}{\partial t}(\mathbf{x},t)dV(\mathbf{x}) = -\int_{\partial\mathcal{D}} Qu_n dS(\mathbf{x}). \tag{1.16}$$

The local expression is then a result of the divergence theorem and the arbitrariness of $\mathcal{D}$.

1.3.1 Conservation of mass

The most basic material property of a fluid is its density $\rho(\mathbf{x},t)$, equal locally to mass per unit volume, here expressed as an Eulerian field. Conservation of mass requires that

$$\frac{d}{dt}\int_{\mathcal{D}_t} \rho dV(\mathbf{x}) = 0, \tag{1.17}$$

or, using (1.12),

$$\frac{d}{dt}\int_{\mathcal{D}_t} \rho dV(\mathbf{x}) = \int_{\mathcal{D}_t} \frac{\partial \rho}{\partial t}(\mathbf{x},t) + \nabla \cdot (\rho \mathbf{u})(\mathbf{x},t) dV(\mathbf{x}) = 0. \tag{1.18}$$

Assuming that the integrand on the right is continuous, this equation applied to arbitrary parcels of fluid implies that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1.19}$$

throughout the domain of fluid. This so-called *equation of continuity* expresses mass conservation as a local constraint on Eulerian variables. The corresponding Lagrangian form of conservation of mass simply states that the total mass contained in a given fluid parcel is a constant. Combining (1.9) nd (1.19) there immediately follows the local equation

$$\frac{d(\rho|\mathbf{J}|)}{dt} = 0, \tag{1.20}$$

see Exercise 1.2. For standard Lagrangian coordinates $\mathbf{J}(\mathbf{a}, 0) = \mathbf{I}$, the identity, and we may then express conservation of mass in the form

$$\rho(\mathbf{a},t)|\mathbf{J}|(\mathbf{a},t) = \rho(\mathbf{a},0). \tag{1.21}$$

if the fluid is incompressible, conservation of mass reduces to the Eulerian statement that density is constant following a fluid particle:

$$\frac{d\rho}{dt} = 0, \tag{1.22}$$

as is clear from (1.19). Density can of course change from particle to particle. A constant density fluid is thus also incompressible, but the converse is not true in general.

1.3.2 Conservation of momentum

In the context of locomotion one must consider both the momentum of the body in locomotion, and of the fluid medium, in order to understand conservation of momentum. For example, consider a body which begins to swim in the absence of gravity, starting from a state of from rest. Since no outside body forces are present, every force exerted by the body on the fluid is accompanied by and equal and opposite force exerted by the fluid on the body. Now Newton's second law states that the rate of change of linear momentum of a mass system with respect to time must equal to the force experienced by the system. Thus if $\mathbf{M}_f$ and $\mathbf{M}_b$ are the total linear momenta of fluid and body respectively, we must have

$$\frac{d}{dt}(\mathbf{M}_b + \mathbf{M}_f) = 0. \tag{1.23}$$

Since the total momentum is constant, and the system starts from rest, $\mathbf{M}_b + \mathbf{M}_f = 0$ for all time. If the fluid fills all space, and the spatial average of velocity over all points of the body is $\mathbf{U}(t)$, the momentum in the fluid at that instant is $-\rho \mathbf{U} V$ where V is the body volume. Note that, if at a given moment the body is accelerating, then there is a balancing change of momentum of the fluid. The body experiences a force, which accelerates it, and the fluid experiences and equal and opposite reaction. If the body stops accelerating, a state that usually involves a constant time average of the body velocity, the average force on the body while swimming in the steady state is zero! This confronts us with the fact that the purpose of locomotion is to displace a body, not to produce a force.

As a second example, consider a heavy, body of weight Win an infinite expanse of fluid. By Archimedes principle, the *relative weight* of the body is W reduced by the buoyancy force $g\rho V$, where $V isthebodyvolume.$

$$W_{rel} = W - g\rho V. \tag{1.24}$$

First suppose that the body deforms in such a way that it can hover at a fixed point. However the hovering may be achieved, the net effect is to apply a force which on average equals the relative body weight W_{rel}. This must in turn equal to the time rate of change of the momentum directed along the gravitational field. In other words, the body must act as a source of downward momentum where $W_{rel} = dM_f/dt$. If the body is rigid and simply sinks, the downward body momentum M_b will increase along with the downward fluid momentum but we will have $W_{rel} = d(M_f + M_b)/dt$.

Within the fluid, Newton's second law states that the time rate of change of momentum $\rho \mathbf{u}$ must equal the force per unit volume $\mathbf{F}$ which acts on the fluid. In Eulerian terms,

$$\rho\Big[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\Big] = \mathbf{F}. \tag{1.25}$$

The corresponding Lagrangian statement is, in the standard variable, just

$$\rho(\mathbf{a}, t)\frac{\partial^2 \mathbf{x}}{\partial t^2}(\mathbf{a}, t) = \rho(\mathbf{a}, 0)\mathbf{J}^{-1} \cdot \frac{\partial^2 \mathbf{x}}{\partial t^2} = \mathbf{F}(\mathbf{a}, t). \tag{1.26}$$

Note that $\frac{\partial^2 \mathbf{x}}{\partial t^2}(\mathbf{a}, t)$ is the fluid acceleration in Lagrangian coordinates. These *local* forms of the conservation of momentum are referred to simply as momentum equations.

The global Eulerian derivation of conservation of momentum again utilizes a fixed domain $\mathcal{D}$ and a flux of momentum through its boundary $\partial\mathcal{D}$. Thus we have

$$\frac{d}{dt}\int_{\mathcal{D}} \rho \mathbf{u} dV(\mathbf{x}) = \int_{\mathcal{D}} \frac{\partial \rho \mathbf{u}}{\partial t} dV(\mathbf{x}) = -\int_{\partial\mathcal{D}} \rho \mathbf{u} u_n dS(\mathbf{x}) + \int_{\mathcal{D}} \mathbf{F} dV(\mathbf{x}). \tag{1.27}$$

The force $\mathbf{F}$ can be divided into a part $\mathbf{f}$ most conveniently described as a body force, and a part which is most conveniently expressed as a surface stress acting on $\partial\mathcal{D}$,

$$\int_{\partial\mathcal{D}} F_i dV(\mathbf{x}) = \int_{\partial\mathcal{D}} f_i dV(\mathbf{x}) + \int_{\partial\mathcal{D}} \sigma_{ij} n_j dS(\mathbf{x}). \tag{1.28}$$

The *stress tensor* σ will be discussed below in the context of the Navier-Stokes equations. The local form of the momentum equation including it may them be writen

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} - \frac{\partial \sigma_{ij}}{\partial x_j} = f_i. \tag{1.29}$$

Not that, if the partial differentiations are carried using the product rule and (1.19) used in the component form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \tag{1.30}$$

then we see that (1.29) is consistent with (1.25),

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{u_i}{\partial x_j} - \frac{\partial \sigma_{ij}}{\partial x_j} = f_i. \tag{1.31}$$

We have till now made no assumptions regarding the fluid except that it has a density field $\rho(\mathbf{x}, t)$ and a velocity field $\mathbf{u}(\mathbf{x}, t)$. The simplest assumption that is made concerning the stress tensor σ is that it be a saclar function multiplied into a unit tensor:

$$\sigma_{ij} = -p(\mathbf{x}t)\delta_{ij}, \quad (\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}. \tag{1.32}$$

Here p is the *pressure field.* . The pressure is an *isotropic* stress field, in the following sense. At any point $\mathbf{x}$ in the fluid where measurements are made, the force on a small surface element $\mathbf{n}dS$, due to a pressure on the side out of which its normal $\mathbf{n}$ points, is exactly $-p(\mathbf{x}, t)\mathbf{n}dS$. That is, the pressure acts in a manner independent of the orientation of the surface. This isotropicity is a direct consequence of Newton's laws. If one considers a small, arbitrary parcel of fluid, and then lets the boundary contract through self-similar shapes, the volume and hence the mass decreases as the length cubed, while the surface area and hence the nominal force which pressure can exert on the parcel decreases as length squared. To prevent divergent acceleration of the parcel, the net pressure force must vanish as the parcel become small. Since the shape is arbitrary the pressure tends to be the same at every point of the surface as its size becomes small.

The very existence of a pressure is an empirical fact. Our fluid continuum has the property that it resists being "squeezed". In general, fluids have the property that as a parcel is squeezed the pressure and density increase. In the case of an incompressible fluid, density is constant and the pressure can be thought of as a resistance to compression which is manifest without perceptible changes of density.

The fluid defined by the choice (1.32) is called an *inviscid, perfect* or *ideal* fluid. The resulting local momentum equation

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \tag{1.33}$$

defines *Euler's equations* of an ideal fluid. With the conservation of mass or continuity equation, we have four equations for the five variables $\rho, p, \mathbf{u}$. A final equation is needed to complete the system. In most of our work here, the final equation will be the stipulation that density is constant. More generally, we might allow p to be an arbitrary function of ρ. In that case we define an *ideal, barotropic* fluid.

1.4 Vorticity and circulation

Locomotion in fluids is usually accompanied by the creation of "eddies", swirling motions of the fluid which are left behind as the body progresses. We show in Figure 1.2 the flow field of a simple flapping wing, as calculated in two-dimensions for a viscous fluid. Eddies are shed from both front and rear edges as the wing moves downward.

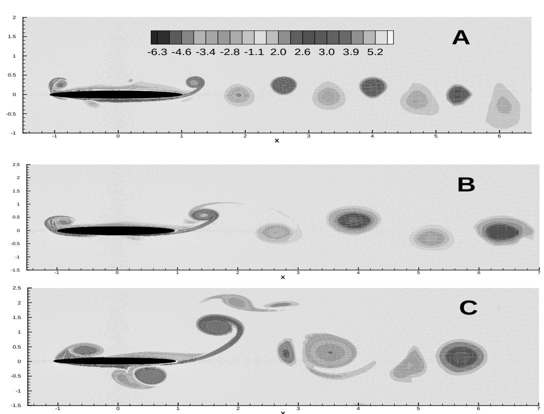

Figure 1.2: Images of the flow field created a thin wing of thin elliptic

cross-section executing sinusoidal flapping motion. Three sequential stages of the downward motion are shown. (Numerical calculation by Jane Wang.)

An important local measure of the intensity of the eddying motion is contained in the *vorticity field*

$$\omega \equiv \nabla \times \mathbf{u}. \tag{1.34}$$

This is here expressed as an Eulerian field, the curl of the velocity field. The vorticity is thus a linear function of the first partial derivatives of the velocity. In general the tensor of first derivatives, $D_{ij} = \frac{\partial u_i}{\partial x_j}$, can be written as a sum of parts symmetric and antisymmetric in the indices:

$$D_{ij} = \frac{1}{2}\big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\big) + \frac{1}{2}\big(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\big) \equiv E_{ij} + \Omega_{ij}. \tag{1.35}$$

Here

$$\mathbf{\Omega} = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & +\omega_1 \\ omega_2 & -\omega_1 & 0 \end{pmatrix} \tag{1.36}$$

Contains the components of the local vorticity. We mention that in two dimensions, that is when $\mathbf{u} = (u(x,y), v(x,y))$, the only surviving component of vorticity is ω_3 and we write $\omega_3 = \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.

Physically, the importance of the vorticity is suggested by the fact that for a rigid body rotation, $\mathbf{u} = \mathbf{w} \times \mathbf{x}$ where $\mathbf{w}$ is a constant vector, we compute $\omega = 2\mathbf{w}$. Now $\mathbf{w}$ is just the angular velocity of the rotation (in radians/sec), and so vorticity is precisely twice the local angular velocity of the fluid. Since conservation of angular *momentum* is important in the dynamics of a fluid, it is then natural to attach a dynamical meaning to vorticity.

This however is not strictly possible, since a massive body refers to a *parcel* of fluid, while vorticity is a point property. If we compute the angular momentum of a parcel P of fluid of density ρ, relative to a point $\mathbf{x}_0$ contained in P, we obtain

$$\mathbf{A}_P \equiv \int_P \rho(\mathbf{x} - \mathbf{x}_0) \times \mathbf{u} dV(\mathbf{x}). \tag{1.37}$$

Let P contain the point $\mathbf{x}_0$ and expand about $\mathbf{x}_0$:

$$\mathbf{A}_P \approx \int_P \rho(\mathbf{x} - \mathbf{x}_0) \times \big(\mathbf{u}_0 + \mathbf{D}_0 \cdot (\mathbf{x} - \mathbf{x}_0)\big) dV(\mathbf{x}). \tag{1.38}$$

Now it is possible to choose P (as an ellipsoid say) and $\mathbf{x}_0$ so that the term in $\mathbf{u}_0$ vanishes, but it is easy to see that in general the remaining term will yield a mixture of contributions from the tensors $\mathbf{E}$ and $\mathbf{\Omega}$, and hence vorticity alone is insufficient as a measure of angular velocity.

It thus turns out that dynamically a more extensive or global measure of "eddying" motion is needed, and this can be obtained by integration of the normal component of the vorticity field over an oriented surface S with boundary ∂S. By Stokes' theorem

$$\int_S \mathbf{n} \cdot \omega dS = \int_{\partial S} \mathbf{u} \cdot d\mathbf{x} \equiv \Gamma_{\partial S}. \tag{1.39}$$

The quantity $\Gamma_{\partial S}$ is the *circulation* of the velocity field $\mathbf{u}$ on the closed material curve ∂S. The value of circulation as a global measure of angular movement is embodied in a fundamental theorem due to Lord Kelvin.

1.4.1 Kelvin's theorem

Consider a fluid with equation of motion (1.25), and suppose that C_t is a simple, closed material curve in the flow. Suppose also that

$$\int_{C_t} \mathbf{F} \cdot d\mathbf{x} = 0. \tag{1.40}$$

Then the circulation Γ_{C_t} of $\mathbf{u}$ on C_t is a constant. To prove this let the curve C_t be a Lagrangian map of a set C_0 of Lagrangian coordinates $\mathbf{a}$. Then

$$\frac{d}{dt}\int_{C_t} \mathbf{u} \cdot d\mathbf{x} = \frac{d}{dt}\int_{C_0} \mathbf{u} \cdot \mathbf{J} \cdot d\mathbf{a}, \tag{1.41}$$

where we bring in the Jacobian $\mathbf{J}$. Thus, since C_0 is independent of time,

$$\begin{aligned} \frac{d}{dt}\int_{C_t} \mathbf{u} \cdot d\mathbf{x} &= \int_{C_0} \left(\frac{\partial \mathbf{u}}{\partial t}\Big|_{\mathbf{a}} \cdot \mathbf{J} + \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial \mathbf{a}} \right) \cdot d\mathbf{a} \\ &= \int_{C_t} \mathbf{F} \cdot d\mathbf{x} + \int_{C_t} \mathbf{u} \cdot d\mathbf{u} = \int_{C_t} \mathbf{F} \cdot d\mathbf{x}, \end{aligned} \tag{1.42}$$

using (1.25), which establishes the result.

Kelvin's theorem is important in fluid mechanics because of the many applications where forces are conservative in the sense that (1.40) holds on all simple closed curves. For example, for an ideal fluid of constant density in a constant gravitational field $\mathbf{g}$ we have $\mathbf{F} = del(\mathbf{g} \cdot \mathbf{x} - \rho^{-1} p)$, and also for an ideal barotropic fluid.

Circulation can be interpreted physically as a measure of "eddying", and so Kelvin's theorem establishes conditions under which eddies are persistent in the flow.

1.4.2 Potential flows

Under what conditions is a flow field eddy–free, in the sense of having zero circulation on all closed contours? A well-known theorem of calculus states that under appropriate smoothness conditions the velocity field must then have the form

$$\mathbf{u} = \nabla\phi \tag{1.43}$$

for some scalar field $\phi(\mathbf{x}, t)$, called the *potential* of the flow. Since $\nabla \times \nabla\phi = 0$ we see that potential flows have no vorticity, that is to say they are *irrotational*.

For a fluid of *constant* density, conservation of mass (see (1.19)) implies that $\nabla \cdot \mathbf{u} = 0$, so if $\mathbf{u}$ is a potential flow the potential must then be harmonic,

$$\nabla^2\phi = 0. \tag{1.44}$$

The theory of harmonic functions tells us that the only harmonic function bounded in $\mathcal{R}^N$ is a constant. The same result holds in the presence of *stationary* boundaries which the potential flow does not penetrate, i.e. where the normal derivative of ϕ vanishes. This suggests that all realistic (e.g. bounded) flows might be expected to be rotational somewhere, and conversely that flows exist because of the vorticity present somewhere. But these statements, while suggestive, are not true in general. A bounded potential *velocity* in infinite space is possible, an example being the uniform flow with $\phi = x$. Also nontrivial potential flows exist in two dimensions in multiply connected bounded domains, and example being $\phi = \theta$ in polar coordinates (r, θ) in the annulus $0 < r_1 < r < r_2$. We say that a domain $\mathcal{D}$ is *simply connected* provided any two continuous curves connecting two points of $\mathcal{D}$ can be deformed continuously one into the other without moving the two points and without leaving $\mathcal{D}$. It then follows (see Exercise 1.8) that there exist no non-zero potential flows in a bounded simply-connected domain. Thus, in such a domain a nonzero flow must be rotational somewhere. In this case we indeed see that the vorticity field is a kind of vortical "skeleton" which supports the flow, and which can in some cases occupy a rather small fraction of the domain.

An extreme example of this is the *point vortex* in two dimensions, defined away from the origin by the potential $\phi = \theta$, or

$$(u, v) = \frac{1}{2\pi} r^{-2}(-y, x). \tag{1.45}$$

A direct calculation of circulation, which may be taken counterclockwise on the circle $r = 1$, gives a value of unity. Since the flow is potential except at the origin, this is the value of circulation on any simple closed curve winding counterclockwise once around the origin. We conclude by appealing to Stokes' theorem that the vorticity $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is concentrated at the origin and has total strength unity. That is, in the language of distributions, $\omega = \delta(x)\delta(y)$. This distribution is then the "skeleton" of the point vortex flow. It is instructive to test ones intuition concerning the meaning of irrotationality on this example. We show in Figure 1.3 the evolution under the flow of a small material parcel. Note that necessarily the parcel is "turning over" as it deforms, which seems to stand in contradiction with the dictum of "zero angular momentum". But we are then trying to compare a point property with a global property. The circulation around the boundary of the parcel remains identically zero. On the other hand, the boundary deforms in response to the symmetric part of the velocity derivative tensor $\mathbf{E}$, see (1.35), and so cannot be related exclusively to the vanishing of vorticity within the parcel (cf. Exercise 1.5).

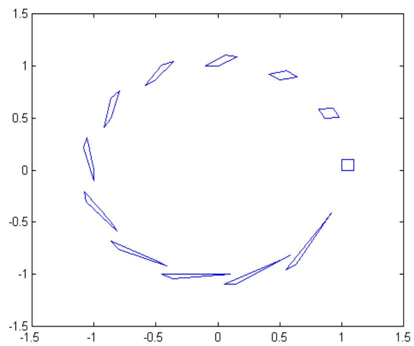

Figure 1.3: Evolution of a small material parcel (here represented by the four corners of a parallelogram) in the point vortex potential flow with $\phi = \theta$.

Potential flows in two dimensions are expecially accessible because of their relation to analytic functions of a complex variable $z = x + iy$. If $f(z) = \phi(x, y) + i\psi(x, y)$ is defined and analytic on a simply-connected, then ϕ is a harmonic function which is a potential of a velocity field (u, v). Moreover $df/dz = f'(z) = u(x, y) - iv(x, y)$. The function $\psi(x, y)$ is the corresponding *stream function* for the flow. This is the harmonic function conjugate to ϕ, satisfying $(u, v) = (\psi_y, -\psi_x)$. The equality $(\phi_x, \phi_y) = (\psi_y, -\psi_x)$ implies the Cauchy-Riemann equations for the function f.

We remark that for the point vortex the associated analytic function is $f(z) = -\frac{i}{2\pi} \log z$, so that $\psi = -\frac{1}{2\pi} \log r$, the *streamlines* ψ=constant being concentric circles.

1.4.3 Velocity from vorticity

Because of the "skeletal" nature of vorticity, it is of interest to try to recover the Eulerian velocity field $\mathbf{u}$ from the vorticity field. Since this means recovering a field from its first derivativess, this is a smoothing operation which can be expressed in integral form. We first note that if, given the vorticity field ω as a function of $\mathbf{x}$, we find a velocity field $\mathbf{v}$ such that $\nabla \times \mathbf{v} = \omega$ then we may add to $\mathbf{v}$ any potential flow and obtain from it the same vorticity. Thus the inversion in question is non-unique up to an added gradient function. We seek to recover the original velocity field $\mathbf{u}$, and the conditions on this field must determine the potential flow.

We suppose $\mathcal{D}$ to be a domain in $\mathcal{R}^3$, and that the domain and the conditions satisfied by the velocity field $\mathbf{u}$ insure uniqueness. For example, if $\mathcal{D}$ is simply-connected and bounded, $u_n = 0$ on $\partial\mathcal{D}$, and both the curl and the divergence of $\mathbf{u}$ are prescribed function, the former being a divergence-free vector field, then uniqueness is obtained.

One such formula for a $\mathbf{v}$, applicable only to certain domain of simple topology is given by the following elegant formula.[2]

$$\mathbf{v} = \int_0^1 s\omega(sx, sy, sz) \times \mathbf{x} ds \tag{1.46}$$

Note that ω must here be defined along rays extending out to the boundary of $\mathcal{D}$ from the origin. In particular then the domain $\mathcal{D}$ must be deformable to a point.

To prove (1.46) we use the vector identity

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla\mathbf{u} - \mathbf{v}\nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla\mathbf{v}, \tag{1.47}$$

to obtain

$$\nabla \times (s\omega(sx, sy, sz) \times \mathbf{x}) = 2s\omega(s\mathbf{x}) + s\mathbf{x} \cdot \nabla\omega = \frac{d}{ds} s^2\omega(s\mathbf{x}), \tag{1.48}$$

so that integration with respect to s yields the desired result. Thus $\mathbf{u}$ and $\mathbf{v}$ have the same curl.

It remains to find a gradient flow $\nabla\phi$ such that $\mathbf{v} + \nabla\phi$ has the same divergence as $\mathbf{u}$. Let the latter be $f(\mathbf{x})$. Taking the divergence of $\mathbf{v}$ we find from (1.46)

$$\nabla \cdot \mathbf{v} = \int_0^1 s\mathbf{x} \cdot (\nabla \times \omega) ds = f - \nabla^2\phi. \tag{1.49}$$

The general solution of this equation is then

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathcal{D}} \Big(\int_0^1 s\mathbf{y} \cdot (\nabla \times \omega(\mathbf{y})) ds - f(\mathbf{y}) \Big) \frac{dV(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} + \phi_h(\mathbf{x}). \tag{1.50}$$

where ϕ_h is an arbitrary function harmonic in $\mathcal{D}$, assumed to be uniquely determined by conditions on $\mathbf{u}$. For recovery of an incompressible flow , we simply set $f = 0$ in the formula.

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[2]This is a special case of the converse to the Poincaré Lemma of the theory of exterior differential forms, see [3]

Although (1.50) provides an explicit solution to the problem at hand, it does not reflect the intuitive association of velocity and vorticity fields which we get from Stokes' theorem. That is, an element of vorticity is associated with circulation according to a right-hand rule. In electromagnetics, the analogous associatiion is between magnetic induction and current, as expressed by the Biot-Savart law. . This provides a different formula for $\mathbf{u}$, which we now discuss in the special case that the domain is $\mathcal{R}^3$ and ω and f vanish sufficiently rapidly at infinity.

If we set $\mathbf{u} = \nabla\phi_f + \mathbf{v}$ where $\nabla^2\phi_f = f(\mathbf{x})$, then again we have the particular solution

$$\phi(\mathbf{x}) = -\frac{1}{4\pi}\int_{\mathcal{R}^3} f(\mathbf{y})\frac{dV(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}. \tag{1.51}$$

We assume the decay of f insures the existance of the this integral for all $\mathbf{x}$ in $\mathcal{R}^3$.

We now seek an incompressible field $\mathbf{v}$ with vorticity ω. We may then introduce a vector potential $\mathbf{A}$ and set $\mathbf{v} = \nabla\times\mathbf{A}$. Assuming tentatively that $\nabla\cdot\mathbf{A} = 0$, and using $\nabla\times(\nabla\times\cdot) = \nabla(\nabla\cdot) - \nabla^2$ we have $-\nabla^2\mathbf{A} = \omega$ and so

$$\mathbf{A} = \frac{1}{4\pi}\int_{\mathcal{R}^3}\omega(\mathbf{y})\frac{dV(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}. \tag{1.52}$$

We then verify that

$$\nabla\cdot\mathbf{A} = \frac{1}{4\pi}\int_{\mathcal{R}^3}\omega(\mathbf{y})\cdot\nabla_{\mathbf{x}}\frac{dV(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} = -\frac{1}{4\pi}\int_{\mathcal{R}^3}\omega(\mathbf{y})\cdot\nabla_{\mathbf{y}}\frac{dV(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} = 0, \tag{1.53}$$

since ω is divergence-free and vanishes at infinity. Thus our flow field has the form $\mathbf{u} = \nabla\times\mathbf{A} + \nabla\phi_f + \nabla\phi_h$ where ϕ_h is a harmonic function. If $\mathbf{u}$ vanishes at infinity then $\nabla\phi_h = 0$. For the case $\nabla\phi_h = f = 0$ the solution has the Biot-Savart form

$$\mathbf{u} = -\frac{1}{4\pi}\int_{\mathcal{R}^3}(\mathbf{x}-\mathbf{y})\times\omega(\mathbf{y})|\mathbf{x}-\mathbf{y}|^{-3}dV(\mathbf{y}). \tag{1.54}$$

The contributions to velocity from the integrand on the right of (1.54) can be understood most easily if ω is thought of as broken to small tubular parcels $\Delta\omega$ with vorticity locally aligned with the axis of the tube. These little vortex tubes each contribute to velocity according to the right-hand-rule: $d\mathbf{u} = -\mathbf{R}\times\Delta\omega R^{-3}$ where $\mathbf{R}$ is the vector from the parcel to the point of evaluation of velocity. This can be useful when discussing the relative effects of isolated patches of vorticity. In general, however, the velocity field asssociated with a continuous distribution of vorticity may be quite difficult to sketch out from this rule.

1.5 The Navier-Stokes equations

1.5.1 The Newtonian stress tensor

In natural fluid dynamics the force $\mathbf{F}$ on the right of (1.25) will contain external force fields such as gravity as well as forces associated with the physical properties of the fluid. We discuss now the fluid properties which will be considered in this monograph. We have introduced above the force exerted by the fluid on a small fluid parcel bounded by an orientable surface, using the surface stress tensor σ_{ij}. We isolate a surface element $\mathbf{n}dS$ of its boundary, and define the force on the element in terms of $\{\sigma_{ij}\}$ by

$$dF_i = \sigma_{ij} n_j dS. \tag{1.55}$$

It is not difficult to show that a divergence of angular momentum on parcels of arbitrarily small volume can be avoided only if σ is a symmetric tensor. The argument is anaologous to that used to show that pressure is an isotropic field. The form of σ_{ij} to be adopted here involves the pressure field $p(\mathbf{x}, t)$ but also now the first derivatives of the velocity field:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu(E_{ij} - \frac{1}{3}\nabla \cdot \mathbf{u}\delta_{ij}), \tag{1.56}$$

where

$$E_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \tag{1.57}$$

is the symmetric part of the velocity derivative tensor.[3] Here μ is the scalar *viscosity* of the fluid. This form of the stress-tensor defines a *Newtonian viscous fluid*. . With the adoption of (1.56) we determine the *Navier-Stokes* momentum equation. . If density is taken as a constant, and if also the viscosity is a constant, then since $\nabla \cdot \mathbf{u} = 0$ the *Navier-Stokes equations* take the form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u} = 0, \tag{1.58}$$

where $\nu = \mu/\rho$ is the *kinematic viscosity* of the fluid. . These form a set of $N + 1$ equations for $\mathbf{u}, p$. In the general compressible case, the equations must be supplemented with an equation of state as well as an equation of conservation of energy.

Once the Navier-Stokes equations are solved subject to suitable boundary conditions (see the following subsection) the stress tensor may be evaluated

[3] The derivation of the form of (1.56) from general principles is given in many textbooks on fluid dynamics, see e.g. [2],[7], [8],[6]. Suffice it to say here that this expression is the simplest one which exhibits the necessary symmetry and does not allow viscous forces to be realized in a rigid-body motion.

to obtain forces. If $\mathbf{n}$ is the normal to a small surface element of area dS adjacent to the fluid flow, then the force exerted by the fluid on the element is $dF_i = \sigma_{ij} n_j dS$. Both pressure and viscous contributions will be important in the problems of interest to us here.

1.5.2 Boundary conditions

Within the scope of Navier-Stokes theory, a fluid is regarded as viscous continuum which cannot move relative to a rigid boundary. We thus apply to any non-fluid boundary a *no-slip condition* . This condition states that at any such boundary the velocity of the fluid is the same as the velocity of a material point on the boundary. If $\mathbf{x}_B(\xi, t)$ denotes points on the boundary, ξ being a vector of parameter determining, for example, the position of all boundary points at time $t = 0$, then no-slip condition states that

$$\mathbf{u}(\mathbf{x}(\xi, t), t) = \frac{\partial \mathbf{x}}{\partial t}(\mathbf{x}, t). \tag{1.59}$$

This assumes that we have prescribed the function $\mathbf{x}(\xi, t)$. In natural locomotion the boundary position will usually not be known in advance, since both current standard shape and the current position of a body will depend upon the fluid flow from initial to current times, see ??.

The non-slip condition is well-verified experimentally for the problems of interest to us here. In sufficiently rarefied gas, on for sufficiently small objects, the assumed continuity of the medium breaks down along with the no-slip condition. In some cases a new effective condition on the continuum is adequate for extending the range of the Navier-Stokes theory. This usually involves a slip at the boundary, that is a non-zero value of the velocity component tangential to the boundary, which depends upon the applied stress field there.

1.5.3 Galilean invariance

Let $\mathbf{U}$ be a constant vector. It is a simple matter to check that if $\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t)$ is a solution of the Navier-Stokes equations, then so is $\mathbf{u}^*(\mathbf{x}^*, t^*), p^*(\mathbf{x}^*, t)$ where

$$\mathbf{u}^*(\mathbf{x}^*, t^*) = \mathbf{u}(\mathbf{x}^*, t^*) + \mathbf{U}, p^*(\mathbf{x}^*, t) + p(\mathbf{x}^*, t^*), \tag{1.60}$$

and

$$\mathbf{x}^*(\mathbf{x}, t) = \mathbf{x} - \mathbf{U}t, t^*(\mathbf{x}, t) = t. \tag{1.61}$$

Indeed, since the stress tensor depends only upon velocity derivatives and contains no time derivatives, it is unaffected by the addition of a constant to velocity or the time dependence of $\mathbf{x}^*$. The material derivative of velocity yields

$$\frac{d\mathbf{u}^*}{dt^*} = \frac{\partial \mathbf{u}^*}{\partial t} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^*, \tag{1.62}$$

where $\nabla^* = \{\frac{\partial}{\partial x_i^*}\}$. Now

$$\frac{\partial \mathbf{u}^*(\mathbf{x}^*, t)}{\partial t} = [\frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} - \mathbf{U} \cdot \nabla \mathbf{u}(\mathbf{x}, t)]_{\mathbf{x}=\mathbf{x}^*}, \tag{1.63}$$

and

$$\mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = [\mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{U} \cdot \nabla \mathbf{u}]_{\mathbf{x}=\mathbf{x}^*}. \tag{1.64}$$

Adding the last two expressions we have invariance of the material derivative and hence therefore of the momentum equation. Invariance of the mass equation follows by a similar argument, and in the case of compressible flow the equations of state and the energy equation may also be shown to be invariant.

Physically, Galilean invariance means that the Navier-Stokes equations govern the velocity observed relative to any frame which moves with uniform velocity. This will be important to us since locomotion with constant velocity is usually most conveniently described relative to a co-moving frame, one that is fixed relative to the body.

1.5.4 The Reynolds number

The Navier-Stokes equations present us with a rather simple dynamical balance in which pressure and viscous forces are balances by the inertial forces associated with the acceleration of the fluid. This reduction of Newton's laws to such a balance indicates the importance of a dimensionless parameter measuring their ratio. The *Reynolds number* may be defined in terms of a length scale L associated with the fluid motions, a characteristic velocity U, and the a typical kinematic viscosity $\nu = \mu/\rho$ where μ is the fluid viscosity. Since a characteristic time is the $T = L/U$, a typical acceleration is U^2/L, leading to inertial forces $\rho U^2 L$. Since two derivatives are involved, viscous forces are of size $\mu_0 U/L^2$. The ratio of inertial to viscous forces is therefore measured by the *Reynolds number*

$$Re = UL/\nu. \tag{1.65}$$

It is important to distinguish the two extremes of small and large Reynolds number. Flows at small Reynolds number are also known as *Stokes flows* or *creeping flows.* . In Stokes flow the inertia of the fluid is negligible, so that the momentum equation reduces to

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \mathbf{F}, \tag{1.66}$$

see (1.56), where on the right we place all other applied forces.

At high Reynolds numbers, on the other hand, viscous forces are *nominally* negligible. The qualifier is important. Suffice it to state here that any neglect of viscous stresses implicitly assumes that the derivatives are not so large as to prevent their neglect. That is, it is not obvious that one has correctly identified the appropriate scales in estimating the stresses as of order $\mu U/L^2$. On a domain where the approximation of negligible viscous stresses may be made, the momentum balance takes the inviscid or Eulerian form noted earlier

$$\rho\big[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\big] + \nabla p = \mathbf{F}. \tag{1.67}$$

The question might be raised as to why there is no still simpler fluid than, say, an incompressible fluid on constant density with momentum equation (1.67). The only available simplification would appear to be that of zero pressure. Although such a model is not without its uses (e.g. in cosmology), if fails to capture an essential feature of everyday fluids, their resistance to compression. An isotropic field in the minimal field to accomplish this, and itg must be retained in any realistic description of most fluids.

In practice it is important to estimate the Reynolds number properly, in order to determine if either of these approximate descriptions is relevant. It is generally a matter of common sense how one should chose U, L. Once dimensionless variables are defined (we do not here give these variables a special symbol) the Reynolds number will appear appropriately in the momentum equation as a dimensionless parameter of the problem. For example the dimensionless analog of the constant-density form of the Navier-Stokes equations is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \frac{1}{Re}\nabla^2\mathbf{u} = 0, \quad \nabla\cdot\mathbf{u} = 0. \tag{1.68}$$

A slight variant of this equation explicitly retains the pressure in the limit of small Reynolds number:

$$Re[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}] + \nabla p^+ - \nu\nabla^2\mathbf{u} = 0, \quad \nabla\cdot\mathbf{u} = 0, \tag{1.69}$$

where $Rep = p^+$. Another way to express this is that an appropriate measure of the pressure forces created in Stokes flow is $\mu U/L$, not ρU^2. We will drop the superscript $+$ when this definition of pressure is used in dimensionless variables.

1.5.5 Frequency as a parameter

In problems of locomotion the body body motion is often essentially periodic, but in any case one can usually assign to a locomoter a typical frequency ω associated with its movements. In the preceding discussion of the dimensionless Navier-Stokes equations we have taken the characteristic time to be L/U, which ties the time scale to the speed of locomotion and body size. The frequency ω must be regarded as an independent parameter, and so a second dimensionless parameter exists. We take this to be the *Strouhal number* St, defined by

$$St = \frac{\omega L}{U}. \tag{1.70}$$

If time is made dimensionless by multiplication by ω instead of U/L, St appears as a multiplier of $\frac{\partial \mathbf{u}}{\partial t}$ in the Navier-Stokes equations.

A derived parameter which will be of interest to us is the *frequency Reynolds number* Re_ω, defined by $Re_\omega = StRe = \omega L^2/\nu$. This parameter is distinguished as as a Reynolds number based entirely upon quantities intrinsic to the locomoter, namely the size and typical frequency, and is independent of whatever speed of locomotion which might result.. An important applicatioin of the frequency Reynolds number is to the *onset* of forward flapping flight. In general both Re_ω and Re are relevant to locomotion, the first characterizing the movements causing locomotion, the second characterizing the consequent speed of locomotion.

1.5.6 gravity

Locomotion in the presence of a gravitational field is of course extremely important for birds and insects, and to a lesser extent for fish and microorganisms. The effects of gravity disappear if the locomoting body has a density equal exactly to that of the fluid, at least provided that a free surface is not present. Otherwise the body can be buoyant and the local acceleration of gravity g is a parameter involved in determining its movements. A dimensionless messure of gravity is usually given as the *Froude number*

$$Fr = \frac{U}{\sqrt{gL}} \tag{1.71}$$

in terms of characteristic velocity and length. This number is a measure of fluid acceleration in units of the acceleration of gravity.

Finally, the mass of the locomoting body, divided by the mass of an equivalent volume of fluid give us a mass ratio r_M

$$r_M = M_{body}/\rho_{fluid} V_{body}. \tag{1.72}$$

1.5.7 Some typical parameter values in Nature

In Table 1 we show some values of Re, Re_ω, and St for various organisms which locomote in the natural fluids of air and water. We retain here the natural division into low and high Reynolds numbers, namely the Stokesian and Eulerian realms introduced earlier. We also indiate an *intermediate realm* separating the two, representing roughly the ragne $1 < Re < 100$. In this range of Reynolds numbers, it is not possible to accurately describe the flow field a dynamic any simpler than the fullk navier-Stokes equations. The intermediate realm is of interest because the Stokesian and Eulerian strategies of locomotion are very different. How then does Nature respond to the ambiguity of intermediate range, where both of the extreme realms are marginally relevant. Not surprisingly, one strategy is to utilize movements wich work in both realms. One example of this is *rowing*, wherein a paddle moves broadside on in a thrusting stroke, and is feathered in a return stroke.

Locomoter		*L(cm)*	*U (cm/sec)*	$\omega(sec^{-1})$	$UL/\nu = Re$	$\omega L/U = St$	$\omega L^2/\nu = Re_\omega$	*Remarks*
Stokesiam	*Bacterium*	10^{-5}	$10^{-2} - 10^{-3}$	10^4	10^{-5}	$10 - 10^2$	$10^{-3} - 10^{-4}$	*Limit of Navier-Stokes theory. Brownian motion affects smaller organisms.*
realm	*Spermatozoan*	$10^{-2} - 10^{-3}$	10^{-2}	10^2	$10^{-2} - 10^{-3}$	$10 - 10^2$	10^{-1}	*Flag. diam.*$\approx 10^{-5} cm.$
	Ciliate	10^{-2}	10^{-1}	10	10^{-1}	1	10^{-1}	*cilium length*$\approx 10^{-3} cm.$
Intermediate	*Small wasp*	10^{-2}	10^{-1}	10	10^{-1}	1	10^{-1}	*U is wing speed hovering*
realm	*Pteropod*	.5	.5	1	25	1	10	*Flapping mode.*
Eulerian	*Locust*	4	400	20	10^4	.2	10^3	*Wing Re* ≈ 2000
realm	*Pigeon*	25	$10^2 - 10^3$	5	10^5	.25	10^4	*Wing Re* $\approx 10^4$
	Fish	50	100	2	5×10^4	1	10^4	

Table 1: Some typical values of Re, St, and Re_ω for various organisms.

One point deserves mention concerning the values of parameters in this table. The range shown for $R4$ and Re_ω is enormous cocmpared to that of St. In fact, in the forward flight of birds as for swimming fish, St tends to lie in the narrow range .2-.4 over a wide range of sizes. One can hope that analysis of the mechanics of locomotion could explain why this is so.

1.6 The geometry of locomotion

1.6.1 Standard and current shape

In the present section we shall consider a very specific example of locomotion in order to introduce some useful geometrical concepts. We consider a problem in two dimensions, a body which is the interior of a smooth deformation of the circle $x^2 + y^2 = 1$, whose shape is determined by an implicit equation $S(x, y, t) = 0$. Let us assume that the body moves in an infinite expanse of fluid along a path determined by the given body motion and the response of the fluid to this motion. The detailed fluid dynamics involved in determining the motion will not be considered in the present discussion. The question is simply this: How should we divide up the description of the overall movement into the "intrinsic" body deformations and the translation and rotation associated with swimming through the fluid? Intuitively, we must describe the "intrinsic" motions by "holding the body fixed". This entails selecting a standard Euclidean frame, and relative to that frame defining the *standard shape* $S_0(x, y, t) = 0$. Then $S(x, y, t) = 0$, which for any value of t we call the *current shape*, must describe both the evolution of the standard shape and the translation and rotation of the Euclidean frame in which the standard shape is defined, relative to its initial position. In Figure 1.4 we show a rotation through angle θ and a translation from $(0, 0)$ to (X, Y) relative to the standard frame.

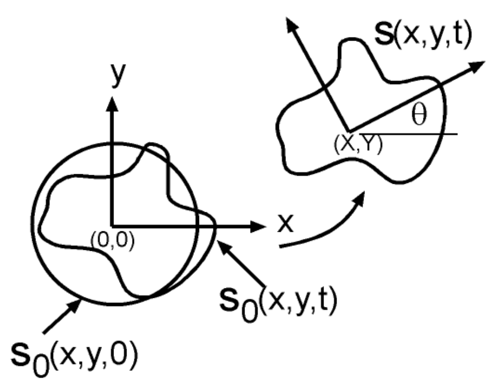

Figure 1.4: The standard shape S_0 at times 0 and t, and the position of the swimming body given by $S(x, y, t)$. At time t the origin of the standard frame has been translated to (X,Y), and it has rotated through an angle θ, giving the current shape at time t.

Now the fluid mechanics will presumably provide a means of computing increments in θ and (X, Y) as the body deforms incrementally. However in general this will depend not only upon the evolution of the standard shape, but also upon the local position determined by (X, Y, θ). This is because the flow field created in the past will generally influence the effect of the current body deformation. To jump to a three-dimensional example from natural flight, a hovering hummingbird moves its wings in the presence of eddies created by previous wing movements. It is not enough to know the incremental movement of the wings at a given time to determine the response of the body to this movement.

What, then, do we mean when we say that this two-dimensional body locomotes? Webster's New Collegiate Dictionary defines locomotion as "Act or power of moving from place to place; progressive movement; hence, travel." It is fair to say that implicit in this definition is the assumption that the power driving the movement comes from the thing which moves (a distinction which applies in the case of a locomotive and an electric tram). The *progressive* nature of locomotion suggests steady movement in a definite direction, and this raises the question of how to deal with a periodic or near-periodic orbit, e.g. one where the current shapes at times t and $t+T$ are the same, and the standard shape has temporal period T. If T were sufficiently small this would not amount to progressive movement, but if it were large enough we might want to consider the closed path as that of a locomoting body.

One case of definite interest can be called *steady locomotion*, by which we shall mean that there is a time T such that

$$X(t+T) = X(t) + \Delta X, \quad Y(t+T) = Y(t) + \Delta X \tag{1.73}$$

for some constants $\Delta X, \Delta Y$ and all times t. We need not consider for the moment what motions might be responsible for the regularity or how θ may vary. We then refer to $(U, V) \equiv (\delta X/T, \Delta Y/T)$ as the *mean velocity of locomotion*. A basic problem is therefore, given a sequence of standard shapes of a body, which repeats itself with some fixed period, to determine if steady locomotion occurs and to determine (U, V) as a function of the shape sequence, once the body is placed in a fluid of known properties.

It is sometimes useful to encode the positional information into a 3×3 matrix

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & X \\ \sin\theta & \cos\theta & Y \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.74}$$

We call $\mathbf{R}$ the *positional matrix*. Note that if x_p, y_p is a point on the standard shape at time t the $\mathbf{R} \cdot v$ locates the point on the current shape $S(x, y, t) = 0$, where v is the column vector $(x_p, y_p, 1)^T$. Suppose now that the current shape is perturbed incrementally from time t to time $t + dt$. The claim is that the matrix

$$\mathbf{A}dt \equiv \mathbf{R}^{-1} \cdot \frac{d\mathbf{R}}{dt} dt \tag{1.75}$$

will be the incremental rigid-body motion relative to the current frame. Indeed, we compute

$$\mathbf{R}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & -X\cos\theta - Y\sin\theta \\ -\sin\theta & \cos\theta & X\sin\theta - Y\cos\theta \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.76}$$

and so

$$\mathbf{A}dt = \begin{pmatrix} 0 & -d\theta & dXcos\theta + dY\sin\theta \\ d\theta & 0 & -dX\sin\theta + dY\cos\theta \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.77}$$

Thus we obtain the correct incremental movement of the point (x_p, y_p) on the standard shape, relative to the current frame:

$$\mathbf{A}\cdot(x_p, y_p, 1)^T dt = (-y_p d\theta + dXcos\theta + dY\sin\theta, x_p d\theta - dX\sin\theta + dY\cos\theta, 0), \tag{1.78}$$

In general the value of $\mathbf{A}$ obtained at a given time will depend upon the prior history of motion, hence upon both $\mathbf{R}$ and the body velocity in the standard frame for preceding times. To find $\mathbf{R}(t)$ we would need to solve $d\mathbf{R}/dt = \mathbf{R}\cdot\mathbf{A}(t)$, assuming the form of $\mathbf{A}$ could be determined given the history of $\mathbf{R}(t)$ and the evolution of the standard shape. In the next Chapter we study the Stokesian realm of locomotion, and it turns out that the situation is much simpler, and the geometry of the standard shape can in fact be very useful to the theory. In the Stokesian realm, $\mathbf{A}$ can be computing at each time instantaneously and directly from the rate of change of the standard shape.

1.7 The basic half-plane problem

To illustrate the solution of the Navier-Stokes equations with a non-slip condition in the simplest setting, we consider a boundary consisting of small deformations of the planar surface $z = 0$. Our goal is to determine the effect of the deformations on the fluid in the half-space $z > 0$.

Since we deal with small deformations, it will be helpful to assume as well that the surface is at all times given by an explicit equation $z_B = f(x, y, t)$, i.e. there is no overturning of the surface. We do allow the boundary to be otherwise arbitrarily deformable as an elastic surface, so that for example it can remain plane while executing tangential stretching as a function of x, y, t. Thus we actually deal with a body deformation described by the vector function

$$\mathbf{x}_B = \mathbf{f}(x, y, t). \tag{1.79}$$

We are interested in a rather general statement of this problem, in order to discuss a number of related issues in locomotion and transport. However the problem is sufficiently difficult analytically to warrant first looking at the simplest special cases.

1.7.1 The swimming of a stretching sheet in Stokes flow

In a seminal paper [10], G.I. Taylor investigated the swimming of a sheet at low Reynolds number. He assumed a fluid of constant viscosity and density, and neglected the inertial terms. He also restricted the problem to two dimensions. Thus the equations for u, v, p are assumed to be

$$\frac{\partial p}{\partial x} - \mu\nabla^2 u = 0, \quad \frac{\partial p}{\partial y} - \mu\nabla^2 v = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.80}$$

To illustrate the calculation we take the special boundary condition

$$x_B = x + a\cos(kx - \omega t), y_B = 0, \tag{1.81}$$

so that the body surface remains plane but is stretched in the x-direction as a progressive wave, the wave-number and frequency of the wave being k, ω. . The boundary conditions on the flow are therefore

$$u(x + a\cos(kx - \omega t), 0, t) = a\omega \sin\xi,\ v(x, 0, t) = 0,\ \xi = kx - \omega t. \tag{1.82}$$

Note that the position at which we impose the condition on u is moving. This introduces a natural expansion in the small dimensionless parameter ak.

We allow for the fact that the dynamic equilibrium of the stretching sheet and the fluid may require that u tends to a finite non-zero number as $y \to +\infty$. (We shall consider here only the upper half-plane.) By Galilean invariance the flow can then be interpreted as the swimming of a sheet calculated relative to a co-moving frame. Thus we suppose

$$u \to U, v \to 0, \text{ as } y \to \infty. \tag{1.83}$$

To satisfy the last of (1.80) we introduce the *streamfunction* $\psi(x, y, t)$, where $u = \psi_y, v = -\psi_x$. Elimination of the pressure from the first two equation in (1.80) shows that ψ is a biharmonic function :

$$\nabla^4\psi = 0. \tag{1.84}$$

Acceptable solutions of (1.84) have the form

$$\psi_m(x, y, t) = (A_m + kB_m y)e^{-mky}\sin m(kx - \omega t)$$
$$+(C_m + kD_m y)e^{-mky}\cos m(kx - \omega t) + U_m y, \tag{1.85}$$

where A_n, B_n, C_n, D_n, U_n are arbitrary constants. Thus ψ will have the form of an infinite series of terms of the form (1.85). We gather these as a series in increasing powers of ak.

The leading term, ψ_1, is or order $\omega k^{-2} \times ak$ and is determined by the conditions

$$\frac{\partial\psi_1}{\partial y}(x, 0, t) = a\omega\sin\xi, \frac{\partial\psi_1}{\partial x}(x, 0, t) = 0. \tag{1.86}$$

Thus we find easily $A_1 = C_1 = D_1 = U_1 = 0$, $kB_1 = a\omega$, so that

$$\psi_1 = ya\omega e^{-ky} \sin\xi. \tag{1.87}$$

Since $U_1 = 0$, it can be said that the sheet does not swim to leading order.

The second-order terms correct for the displaced location of the boundary condition on u:

$$\begin{aligned} u(x + a\cos\xi, 0, t) &= u_1(x, 0, t) + \frac{\partial u_1}{\partial x}(x, 0, t) \times a\cos\xi + u_2(x, 0, t)\ldots \\ &= a\omega \sin\xi. \end{aligned} \tag{1.88}$$

Thus u_2, v_2 satisfy the conditions

$$u_2(x, 0, t) = -\frac{\partial u_1}{\partial x}(x, 0, t) \times a\cos\xi, v_2(x, 0, t) = 0. \tag{1.89}$$

The right-hand-side now involves a $\cos^2(\xi)$and so the corresponding term ψ_2 takes the form

$$\psi_2 = -\frac{1}{2}\omega k a^2 y e^{-2ky} \cos 2\xi - \frac{1}{2}\omega k a^2 y. \tag{1.90}$$

Thus $U_2 = -\frac{1}{2}ka^2\omega$ determines the leading term of the expansion of the swimming velocity. Assuming $k, \omega > 0$ we see that the flow at infinity is in the direction of negative x, so that the swimming velocity has the same sign as the velocity of propagation of the wave of stretching. The physical reason for this lies in the effect of the boundary conditions on the eddy structure near the sheet. The propagating wave of stretching introduces an asymmetry into the eddy pattern, as we display in Figure 1.5.

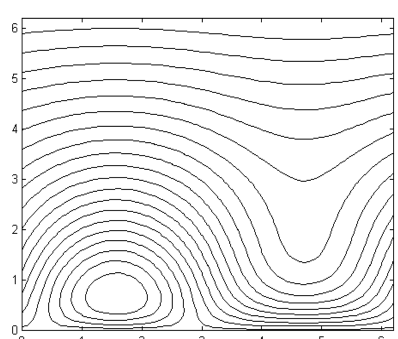

Figure 1.5: $\psi(x, y, t)$, in coordinates $(\xi, \eta) = (kx - \omega t, ky)$, for a stretching plane sheet in Stokes flow, with $k = \omega = 1, a = .3$.

The flow along the top of the figure is from right to left. Note that the flow near the wall is dominated by motion in this direction. The counter-rotating eddy on the left exerts a thrust to the right at the wall, but this is more than compensating by the dominant thrust to the left. The reaction of this net force on the fluid is what drives the sheet through the fluid.

The last remark emphasized our calculation as one of swimming. On the other hand we might just as well have stated that the effect of the sheet is to drive a uniform flow extending to infinity! This seems paradoxical, that a sheet could drive an infinite expanse of fluid into uniform motion. The resolution of this lies in the fact that the sheet is also infinite in expanse. Any finite piece of the sheet would have not such an effect at infinity, although it would swim relative to the fluid at infinity. The speed of swimming would presumably be altered somewhat by the finite size.

We draw attention as well to the relation of our calculation to a related estimate of pumping in a channel with flexible walls. It is reasonable to assume that in a channel of width $H >> k^{-1}$, with two walls executing the movements considered here, fluid would be pumped with a velocity given approximately by U_2.

1.7.2 Finite and large Reynolds numbers

We now consider the problem of the preceding subsection with no special assumption on the Reynolds number, still retaining the same form and small amplitude of stretching. It is instructive now to adopt a dimensionless formulation.[4] If $U = \omega/k$, $L = 1/k$, T$= 1/\omega$ are the reference velocity, length, and time, the dimensionless momentum equations are now

$$Re[-\frac{\partial}{\partial\xi} + u\frac{\partial}{\partial\xi} + v\frac{\partial}{\partial\eta}](u, v) + (p_\xi, p_\eta) - \nabla^2(u, v) = 0, \tag{1.91}$$

where $\eta = ky$ and as before $\xi = kx - \omega t$, ∇^2 now being in the latter variables. We have assumed here, based upon the boundary conditions (1.81) that the variable depends only upon ξ, η. Since velocities remain small, of amplitude $\epsilon \equiv ak$, the expansion of the dimensionless streamfunction ψ is

$$\psi(\xi, \eta) = \epsilon\psi_1(\xi, \eta) + \epsilon^2\psi_2(\xi, \eta) + \dots, \tag{1.92}$$

where, after eliminating pressure

$$Re\frac{\partial\nabla^2\psi_1}{\partial\xi} + \nabla^4\psi_1 \equiv L\psi_1 = 0. \tag{1.93}$$

[4]The formulation of physical problems in dimensionless variables usually has the advantage of clarifying the parametric structure of a problem. By clearly exhibiting the sizes of dimensionless parameters, it is far easier to arrive at useful and appropriate simplifications of the problem. On the other hand the dimensionless forms expel useful checks on algebra made possible by the dimensional consistency that must be maintained in the mathematical equations.

We see that we may write

$$\psi_1 = \Re[Ae^{-\eta+i\xi} + Be^{-\lambda\eta+i\xi}], \quad \lambda = \sqrt{1 - iRe}. \tag{1.94}$$

The boundary conditions (1.86) then yield

$$\psi_1 = Re^{-1}\Re[(\lambda + 1)(e^{-\eta+i\xi} - e^{-\lambda\eta+i\xi})]. \tag{1.95}$$

For ψ_2 we now have two contributions $\psi_2 = \psi_{21} + \psi_{22}$. The first comes from the shifted boundary condition and is obtained as in the case of Stokes flow:

$$\psi_{21} = -\frac{1}{4}Re^{-1}\Re[(e^{-2\eta} - e^{-2\lambda\eta})e^{2i\xi}(1 + \lambda)] - \eta/2. \tag{1.96}$$

The second contribution comes from the nonlinear terms and satisfies null conditions on $\eta = 0$. The equation satisfied by ψ_{22} is

$$L\psi_{22} = -Re\frac{\partial(\psi_1, \nabla^2\psi_1)}{\partial(\xi, \eta)}. \tag{1.97}$$

Since we will focus on the swimming velocity, we take the ξ-average of (1.97). With $\langle\cdot\rangle$ denoting this average, using (1.94) we obtain

$$\frac{d^4\langle\psi_{22}\rangle}{d\eta^4} = -\frac{Re^2}{2}\Re[AB^*(1+\lambda^*)e^{-(1+\lambda^*)\eta} + |B|^2(\lambda+\lambda^*)e^{-(\lambda+\lambda^*)\eta}]. \tag{1.98}$$

Integrating three times with respect to η and using the conditions at infinity, we have

$$\langle u_{22}\rangle(0) = U_{22} + \frac{Re^2}{2}\Re[AB^*(1 + \lambda^*)^{-2} + |B|^2(\lambda + \lambda^*)^{-2}] = 0. \tag{1.99}$$

Since $A = -B = (1 + \lambda)/Re$,

$$U_{22} = \frac{1}{2}\Re\Big[\frac{1 + \lambda}{1 + \lambda^*} - \frac{|1 + \lambda|^2}{(\lambda + \lambda^*)^2}\Big]. \tag{1.100}$$

Since $\Re(\lambda) = F(Re) \equiv [(1 + \sqrt{1 + Re^2})/2]^{1/2}$, this expression can be reduced to

$$U_{22} = \frac{1}{4}[1/F - 1]. \tag{1.101}$$

But $U_{21} = -1/2$, see (1.96), so we arrive at the swimming velocity

$$U = \epsilon^2 U_2 = \epsilon^2\frac{1}{4}[1/F - 3]. \tag{1.102}$$

We thus see that as Re increases from 0 to ∞ the swimming speed increases from our earlier result of 1/2, but remains finite and equal to 3/4 in the limit.[5]

[5] In dimensional variables, the streamfunction acquires the prefactor ω/k^2 so (1.102) takes the form (see [11]) $U = a^2k\omega\frac{1}{4}[1/F - 3]$.

We have considered this calculation early in our study, because it highlights the problems we face with the inviscid limit $Re \to \infty$. Eulerian fluid dynamics generally drops the non-slip condition as inapplicable to a fluid with zero viscosity, sometimes referred to as a "slippery" fluid.[6] The natural boundary conditions are those requiring the fluid to have zero normal component of velocity relative to any rigid surface. In the present case of a stretching plane surface, the natural boundary condition on the fluid is simply that $v = 0$ on the wall. This leaves the tangential component free, and for a plane wall $y = 0$ the motion in the x-direction is indeterminant. One can also say that the tangential motions associated with the stretching sheet should have no effect on the fluid. Our calculation shows that this inviscid result to not the same as the inviscid *limit* of the viscous flow. The dynamic balance imposed at finite Reynolds number is retained in the limit. This distinction between the inviscid theory and the limit for large Reynolds number will be a recurring theme of our investigations. In effect we see that both locomotion and transport can be subject to this paradoxical property of the inviscid limit. This is problem often referred to as a *non-uniformity* of the (inviscid) limit.

How can we understand this non-uniformity in the present example? The answer must lie in the boundary-layer structure of the flow once Re become large. We can see this structure in the appearance of the factor $e^{-\lambda\eta}$. The issue after all is this: can we have a non-zero uniform flow adjacent to a planar surface where the no-slip condition is satisfied, such that there is no average x-component of force on the surface? The solution in Stokes flow answers this in the affirmative. In that case the viscous stresses dominate inertia, so this is a very essential dynamical balance.

In the limit of large Re, The viscous stress tensor has, in dimensionless variables, a prefactor Re^{-1}. The factor $e^{-\lambda\eta}$ indicates a variation near $\eta = 0$ on the scale λ^{-1}, of order $Re^{-1/2}$, and since u is O(1) the velocity derivatives are of order $Re^{1/2}$. Thus the viscous force associated with the boundary layer is actually small, of order $Re^{-1/2}$. (Note that the O(1) scale in η gives an even smaller contribution to the viscous force.) These forces tend to zero under the inviscid limit as they should, but nevertheless the free-stream velocity for dynamical equilibrium can and does remain finite. To put this in another way, in the limit the sheet slides through the fluid easily, so the swimming speed can be decided by the small forces developed in the boundary layer.

We have thus shown that the sheet "can swim in an inviscid fluid", and determined the speed, but the statement only has meaning when in-

[6] Such "ideal" fluids are never exactly obtained at ordinary temperatures and pressures, but are approached in the superfluid component of liquid helium. The relevance of the inviscid theory to naturally occurring flows, well outside of boundary layers, is an important classical problem which is discussed in the standard textbooks.

terpreted as a limit. We add that a more generalized half-plane problem involving out-of-plane deformations leads to similar conclusions and will be discussed in section ??.

1.8 Exercises

1.1. The determinant of $\mathbf{J}$ may be defined by

$$|\mathbf{J}| = \sum \varepsilon_{ijk} \frac{\partial x_1}{\partial a_i} \frac{\partial x_2}{\partial a_j} \frac{\partial x_3}{\partial a_k}.$$

Here $\varepsilon_{ijk} = 1$ if ijk is an even permutation of 123, $= -1$ if an odd permutation, and is zero otherwise. Note also that

$$\sum \varepsilon_{ijk} \frac{\partial x_m}{\partial a_i} \frac{\partial x_2}{\partial a_j} \frac{\partial x_3}{\partial a_k} = 0, m = 2, 3.$$

Use these facts to establish (1.9).

1.2. Verify (1.20) by differentiating an integral over a parcel. Alternatively obtain it by combining (1.9) and (1.19).

1.3. In one dimension, the Eulerian velocity is given to be $u(x,t) = 2x/(1+t)$. (a) Find the Lagrangian coordinate $x(a,t)$. (b) Find the Lagrangian velocity as a function of a, t. (c) Find the Jacobian $\partial x/\partial a = J$ as a function of a, t. (d) If density satisfies $\rho(x,0) = x$ and mass is conserved, find $\rho(a,t)$ using the Lagrangian form of mass conservation. (e) From (a) and (d) evaluate ρ as a function of x, t, and verify that the Eulerian conservation of mass equation is satisfied by $\rho(x,t), u(x,t)$.

1.4. Let $\mathcal{D}_t$ denote a fluid volume in three-dimensions. Prove that, for any smooth function $\mathbf{g}(\mathbf{x}, t)$,

$$\frac{d}{dt} \int_{\mathcal{D}_t} \rho \mathbf{g} \, dV(\mathbf{x}) = \int_{\mathcal{D}_t} \rho d\mathbf{g}/dt \, dV(\mathbf{x}).$$

Here ρ is the density, satisfying the mass conservation equation $\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$, and $d/dt = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ is the material derivative.

1.5. Consider the "point vortex " flow in two dimensions,

$$(u, v) = UL(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}), \ x^2 + y^2 \neq 0,$$

where U, L are reference values of speed and length. (a) Show that the Lagrangian coordinates for this flow may be written

$$x(a,b,t) = R_0 \cos(\omega t + \theta_0), \ y(a,b,t) = R_0 \sin(\omega t + \theta_0)$$

where $R_0^2 = a^2 + b^2, \theta_0 = \arctan(b/a)$, and $\omega = UL/R_0^2$. (b) Consider, at $t = 0$ a small rectangle of marked fluid particles determined by the points $A(L, -\Delta y/2), B(L + \Delta x, -\Delta y/2), C(L + \Delta x, \Delta y/2), D(L, \Delta y/2)$. If the points move with the fluid, once point A returns to its initial position what is the shape of the marked region? Since $(\Delta x, \Delta y)$ are small, you may assume the region remains a parallelogram. Do this, first, by computing the entry $\partial y/\partial a$ in the Jacobian, evaluated at $A(L, 0)$. Then verify your result by considering the "lag" of particle B as it moves on a slightly larger circle at a slightly slower speed, relative to particle A, for a time taken by A to complete one revolution.

1.6. Lagrangian coordinates can use any unique labeling of fluid particles. To illustrate this, consider the Lagrangian coordinates in two dimensions

$$x(a,b,t) = a + \frac{1}{k}e^{kb}\sin k(a+ct),\ y = b - \frac{1}{k}e^{kb}\cos k(a+ct),$$

where k, c are constants. Note here $a, b \neq 0$ are *not* the initial coordinates. By examining the determinant of the Jacobian, verify that this gives a unique labeling of fluid particles. (These waves, which were discovered by Gerstner in 1802, represent gravity waves if $c^2 = g/k$ where g is the acceleration of gravity. They do not have any simple Eulerian representation.)

1.7. Consider the two-dimensional Eulerian flow $(u,v) = U/L(x,-y)$. Show that a fluid particle in the first quadrant which crosses the line $y = L$ at time $t = 0$, crosses the line $x = L$ at time $t = \frac{L}{U}\log{(UL/\psi)}$ on the streamline $Uxy/L = \psi$. Do this two ways. First, consider a line integral of $\mathbf{u} \cdot \vec{ds}/(u^2+v^2)$ along a streamline. Then compute differently using Lagrangian variables.

1.8. Prove that any potential flow in a bounded, simply connected domain must vanish, provided the normal velocity component vanishes at the boundary.

1.9. (Non-existence of large hummingbirds). Because of stress limitation on bones, it is known that the power available for the hovering of birds is proportion to L^2, where L is a typical length representing the size of the bird. Show that the power *required* for hovering is proportional to $L^{7/2}$. Assume bird weight proportional to L^3. Use the fact that the required power is the speed U of the downward jet created in hovering times the force needed to hover. Assume the downward jet area is proportional to L^2, and consider the momentum it carries.

1.10. Find a solution of the Navier-Stokes equations for a flow in two dimensions of the form $\mathbf{u} = (u(y,t),0)$, $p = 0$, in the domaian bounded by $y = 0, H$. The surface $y = 0$ oscillates sinusoidally with amplitude A and frequency ω. The upper surface $y = H$ is held fixed. Because of the no-slip condition, we must have $u(0,t) = \omega A\cos(\omega t)$, and $u(H,t) = 0$. Show that $u(y,t)$ satisfies the heat equation $u_t - \nu u_{yy} = 0$ and solve for $u(y,t)$ by separation of variables. Compute the force (per unit area) on the two walls, and the momentum per unit area of the fluid, and verify Newton's law (force on fluid equals time derivative of total momentum, all per unit area).

1.11. Define the complex derivatives

$$\frac{d}{dz} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big),\ \frac{d}{d\bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big).$$

Applied to a complex-valued function $w = F(x,y) + iG(x,y)$, where f, g

are very smooth, verify that

(a) $\frac{dw}{d\bar{z}} = 0 \Rightarrow \ \nabla^2 F = 0, \nabla^2 G = 0;$

(b)$\frac{d^2 w}{d\bar{z}^2} = 0 \Rightarrow \ \nabla^4 F = 0, \nabla^4 G = 0;$

Also show that $\frac{df}{d\bar{z}} = 0$ for any analytic function of the complex variable $z = x + iy$, and that $\frac{df(\bar{z}}{d\bar{z}}) = f'(\bar{z})$. Thus show that general solutions of the biharmonic equation $\nabla^4 \psi = 0$ in two dimensions are provided by the real and imaginary parts of $\bar{z}f(z) + g(z)$, where f, g are analytic functions of a complex variable. Relate this general result to the particular solutions $(A + By)e^{\pm kx})(\sin kx, \cos kx)$ used in the study of the swimming sheet.

1.12. What is wrong with the following reasoning? In the planar, stretching swimming sheet, the plane just oscillates back and forth, just like the the case of an oscillating plane wall. But we can formulat the latter problem as follows: we have $u, v = (u(y,t), 0)$, with $u_t - \nu u_{yy} = 0$, and conditions $u(\infty, t) = 0, u(0,t) = Acos(\omega t)$. The solution is easily seen to be *exactly*

$$u = Ae^{-y/over2\sqrt{\nu}} cos(\omega t - y/over2\sqrt{\nu}.$$

The flow decayed exactly to zero exponentially in y. The same thing should happen for the stretching flat sheet sheet, so in fact it cannot swim!

References

[1] George Backus. A class of self-sustaining dissipative dynamos. *Annals of Physics*, 4, 1958.

[2] G.K. Batchlor. *An Introduction to Fluid Mechanics.* Cambridge University Press, 1967.

[3] Harley Flanders. *Differential Forms.* Academic Press, New York, 1963.

[4] John Happel and Howard Brenner. *Low Reynolds Number Hydrodynamics.* Prentice-Hall. Inc., Englewood Cliffs, N.J., 1965.

[5] Q.A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow.* Gordon and Breach, New York, 1963.

[6] Sir Horace Lamb. *Hydrodynamics.* Cambridge University Press, sixth edition, 1932.

[7] L.D. Landau and E.M. Lifshitz. *Fluid Mechanics*, volume 6 of *Course in Theoretical Physics.* Pergammon Press, New York, second edition, 1987.

[8] L.M. Milne-Thomson. *Theoretical Hydrodynamics.* The Macmillan Company, New York, third edition, 1955.

[9] P.H. Roberts. *An Introduction to Magnetohydrodynamics.* American Elsevier Publishing Company, Inc., New York, 1967.

[10] G.I. Taylor. Analysis of the swimming of microscopic organisms. *Proc. Roy. Soc. Lond.*, A209:447–461, 1951.

[11] E.O. Tuck. A note on the swimming problem. *Jour. Fluid Mech.*, 31:305–308, 1968.

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