3

Eulerian locomotion

3.1 Inertial forces

We turn now to the theory of high Reynolds number flows, and the general setting for addressing fluid locomotion of the larger animals. returning to the dimensional form of the navier-Stokes equations for a fluid of constant density in a uniform gravitational field, we have

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p - \nu \nabla^2 \mathbf{u} = \mathbf{g}, \ \nabla \cdot \mathbf{u} = 0. \tag{3.1}$$

We now wish to formally let $\nu \to 0$ so as to remove viscous stressed from consideration. We have seen in Chapter 1 that this limit process is a singular one, owing to the importance of viscous stresses in the neighborhood of surfaces where a no-slip condition applies and boundary layer form. In the present section we shall disregard all such complications and treat the fluid as an *inviscid fluid* (the terms *ideal fluid* or *perfect fluid* are also used), possessing no viscosity and therefore freely sliding over boundary surfaces.

Although the ideal fluid assumption is very suspect in most applications involving boundaries, there is a certain justification in this approximation in the nonstationary problems frequently encountered in the problems of locomotion. This is because the effects of viscosity take some time to become manifest in a fluid, and if the motion of a body is cyclic and sufficiently rapid, deviation from the ideal theory may be minimized. Nevertheless, the reader should understand that all inviscid modeling needs to be carefully assessed , by comparison with experimental observations and with viscous modeling based upon the full Navier-Stokes equations.

We will thus consider the *Euler equations* for an inviscid fluid of constant density,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p = \mathbf{g}, \ \nabla \cdot \mathbf{u} = 0. \tag{3.2}$$

The Euler system is minimal for fluid dynamics. Since we can write the momentum equationa as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p* = 0, \ p* = p/\rho - \mathbf{g} \cdot \mathbf{r}, \tag{3.3}$$

we see that no parameters remain, and the nonlinearity of the fluid equations embodied in the inertial term $\mathbf{u} \cdot \mathbf{u}$ is an inescapable fact. (We will however avail ourselves of geometrical assumptions, an example being the consideration of bodies which are thin or slender, which allow simplifications in the treatment of the nonlinearity.

Propulsive mechanisms based upon Euler's equations depend fundamentally upon the *reaction* of the fluid to accelerations of the body. This principle basis of locomotion within the Eulerian realm is totally different from the viscous *resistive* forces of the Stokesian realm. The terms

$$\rho \frac{d\mathbf{u}}{dt} \equiv \rho\big(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\big), \tag{3.4}$$

equal to the force needed to accelerate the fluid per unit volume, fully describes this reaction.

It is revealing to adopt a somewhat unconventional and somewhat old-fashioned terminology, based upon the vector identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla(\frac{1}{2}u^2) - \mathbf{u} \times \nabla \times \mathbf{u} \tag{3.5}$$

in (3.4). The term

$$\rho\big(\frac{\partial \mathbf{u}}{\partial t} + \nabla(\frac{1}{2}u^2)\big) \tag{3.6}$$

will be termed the *inertial force*. The remaining term,

$$-\rho \mathbf{u} \times \nabla \times \mathbf{u} \tag{3.7}$$

will be termed the *vortex force*, making reference to the vorticity $\omega = \nabla \times \mathbf{u}$. Strictly speaking both of these forces are inertial, but the special proerties of the vorticity field, and the fact that in many flows vorticity is largely absent, make this a useful way to split up the nonlinear term.

The concern of the present section is the inertial force, so we shall set $\omega = 0$ and assume potential flow $\mathbf{u} = \nabla \phi$, where the potential ϕ necessarily satisfies Lapalce's equation,

$$\nabla^2 \phi = 0, \tag{3.8}$$

in th3 domain occupied by the fluid. Note that ϕ depends in general upon both $\mathbf{r}$ and t.

3.1.1 Boundary conditions

At a moving but solid boundary immersed in an ideal fluid, we must relax the condition of strict adherance of the fluid. However, if the boundary is impenetrable to the fluid, as we shall alwasy assume here, the fluid on the boundary is allowed to move tangentially along it. Specifically, if the implicit equation $S(\mathbf{r}, t) = 0$ specifies the body, surface $B(t)$, The S must be a material invariant in the Lagrangian sense, that is to say invariant on the trajectory of any fluid particle. therefore

$$\frac{dS}{dt}\Big|_B = \Big[\frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S\Big]_B = 0. \tag{3.9}$$

Note that on *stationary* surfaces, the time derivative drops out and the condition is that the component of velocity normal to the surface must vanish.

3.1.2 The unsteady Bernoulli theorem

We consider irrotational unsteady flow. The momentum equation then reduces to

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla(\frac{1}{2}u^2) + \frac{p}{\rho} = \mathbf{g}, \tag{3.10}$$

which can be written

$$\nabla\big[\phi_t + \frac{1}{2}(\nabla\phi)^2 - \mathbf{g} \cdot \mathbf{r}\big] = 0. \tag{3.11}$$

Thus

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} - \mathbf{g} \cdot \mathbf{r} = f(t). \tag{3.12}$$

Generally one takes $f = 0$ since $\phi - f'$ can be redefined as the potential without affecting $\mathbf{u}$ or the calculation of force on a body. The importance of (3.12) is as the equation of pressure in an unsteady flow, allowing the calculation of the pressure (here inertial) force.

3.1.3 The moving cylinder

As an example of a force calculation using the unsteady Bernoulli theorem, consider a circular cylinder of radius a, moving along the x-axis through a fluid otherwise at rest, with velocity $(U(t), 0, 0)$, no gravity. The boundary conditition at the cylinder can then be obtained from the implicit equation $S = (x - X(t))^2 + y^2 = 0,\ \dot{X} = U$. Then

$$\frac{dS}{dt} = -2U(x - X) + 2((x - X)u + 2yv = 0 \tag{3.13}$$

or, at the instant when the center of the cylinder is at the origin,

$$u_r = \frac{\partial \phi}{\partial r} = U \cos\theta \tag{3.14}$$

when $r = a$. We require that ϕ vanish at infinity. The harmonic function which satisfies these conditions is

$$\phi = -a^2 U(t) r^{-1} \cos\theta. \tag{3.15}$$

The velocoty components and pressure are then

$$u_r = a^2 U r^{-2} \cos\theta,\ u_\theta = a^2 U r^{-2} \sin\theta, \tag{3.16}$$

$$\frac{p}{\rho} = a^2 \dot{U} r^{-1} \cos\theta - \frac{1}{2} a^4 r^{-4} U^2. \tag{3.17}$$

The second term on the right of (3.17) is independentof θ, at at the cylinder surface exerts no net force. Integrating $\cos\theta p$ around $r = a$, we obtain the force exerted by the cylinder on the fluid,

$$F = \rho a^2 \pi \dot{U}. \tag{3.18}$$

This force can be thought of as the inertial reaction of the fluid which is set in motion by the movement of the cylinder, It is in the form $F = ma$ of Newton's law of motion with $a = \dot{U}$ and $m = \rho\pi a^2$. This *apparent* or *virtual* or *effective* mass comes from the density of the fluid. It has nothing to do with what mass may or may not be associated with the material of the cylinder itself. (The latter must of course be included if one is to study the movement of the cylinder in response to an applied force.) Curiously, the apparent mass is here equal to the the mass per unit length of the fluid that is displaced by the cylinder. This is coincidental. A flat plate displaces not fluid, but intuition suggests that it will have an apparent mass when moved broadside on, but none when moved tangentially. This last observation suggests that actually apparent mass is a tensor quantity. We shall verify this property below.

Another way to view the phenomenon of apparent mass is in terms of energy. When we set the cylinder in motion we create kinetic energy in the fluid. From (**??**) we compute this energy as

$$E = \frac{1}{2} 2\pi\rho a^4 \int_0^\infty r^{-3} dr = \frac{\pi}{2} \rho a^2. \tag{3.19}$$

The force $F = m\dot{U}$ which accelerates the cylinderfor rest to velocity U does work $W = \frac{m}{2} U^2$, and by conservatiion of energy, $W = E$, again yielding $m = \rho a^2 \pi$. Note that if a body is in oscillation, the cylinder does work on on the fluid when it accelerates and the kinetic energy increases, but work is done on the body as it decelerates and kinetic energy decreases.

3.1.4 Translation of an arbitrary rigid body

By translation of a rigid body, we mean that every point of the body moves with the same velocity $\mathbf{U}(t)$. Now in irrotational flow the boundary condition (3.14) becomes for a general body with outward normal $\mathbf{n}$, relative to the stationary observer who see the fluid at infinity at rest,

$$\frac{\partial \phi}{\partial n} = \mathbf{U} \cdot \mathbf{n},\ \mathbf{r} \in S. \tag{3.20}$$

We consider flows in both two (N=2) and three (N=3) dimensions. The kinetic energy of the fluid is defined by

$$E(t) = \frac{1}{2}\rho \int_V (\nabla\phi)^2 dV, \tag{3.21}$$

where V is the domain exterior to the body B. We shall suppose that this improper integral is convergent. The slowest decays harmonic components develope by a finite body in translation in fact are the dipoles, for which the decay of ϕ is like r^{1-N}, which is more than sufficient for (3.21) to exist. Note that for a body translating with constant velocity, this energy is itself independent of time

Since both Laplaces equation and (3.20) are linear in ϕ, we may write ϕ in terms of a vector function Φ,

$$\phi = \mathbf{U} \cdot \Phi, \tag{3.22}$$

where

$$\mathbf{n} \cdot \nabla\Phi = \mathbf{n},\ \mathbf{r} \in S. \tag{3.23}$$

Since Φ depends only on the instantaneous shape of the body.

Now the differential form of conservation of energy is

$$\frac{dE}{dt} = \mathbf{F}(t) \cdot \mathbf{U}(t), \tag{3.24}$$

where $\mathbf{F}$ is the force exerted by the body on the fluid.

D'Alembert's paradox

From (3.24) it follows that a body translating with constant velocity must have E =constant and so $\mathbf{F} \cdot \mathbf{U} = 0$. The component of the force exerted by body on the fluid, in the direction of the velocity of the body, is usually referred to as the body *drag*. D'Alembert's paradox is thus that this drag is zero in steady translation of a body from rest through an inviscid fluid. Of course, because of the special properties of an inviscid fluid, particularly near boundaries, it is not surprising that that we can deduce a very nonphysical result.

What about other features uniform translation? In general a nonzero *torque* is needed to sustain a body in uniform translation. Also, in two dimensions the force orthogonal to $\mathbf{U}$, called *lift* when this direction is opposing gravity, need not vanish see section ??. This is a result of the 2D exterior domain failing to be simply connected. But in three dimensions, $\mathbf{F} = 0$ in its entirety in irrotational flow, see section ??.

Apparent mass

One way to approach the computation of apparent mass would be to comput the total momentum of the fluid

$$\rho \int_V \nabla\phi dV. \tag{3.25}$$

The rate of change of this momentum should be apparent mass time acceleration. Unfortunately, the $O(r^{1-N})$ decay of ϕ leaves the improper integral conditionally convergent. To avoid this difficulty, we work with the kinetic energy. Consider the regiion V between the body surface S and a large, distant spehreical surface Σ. With ϕ the potential of the flow seen by the stationary observer, we consider

$$E == \frac{1}{2}\rho \int_V (\nabla\phi)^2 dV$$

$$= -\frac{1}{2}\rho \int_V (\mathbf{U} + \nabla\phi)(\mathbf{U} - \nabla\phi) dV + \frac{1}{2}U^2 J(V), \tag{3.26}$$

where $J(V)$ is the content (area or volume) of V. Now by (3.20) $\mathbf{U} - \nabla\phi$ has zero normal component on S (a fact that is the main motivation for the method). Thus, by writing the integrand in (3.26) as $\nabla\cdot[(\mathbf{U}\cdot\mathbf{r}+\phi)(\mathbf{U}-\nabla\phi)]$ and using the divergence theorem,

$$E = \frac{1}{2}\rho \int_\Sigma \Big(\frac{\partial\phi}{\partial n} - \mathbf{U}\cdot\mathbf{n}\Big)(\mathbf{U}\cdot\mathbf{r} + \phi) d\Sigma + \frac{1}{2}U^2 J(V). \tag{3.27}$$

Let

$$\phi = -\frac{\mathbf{A}(t)\cdot\mathbf{r}}{r^N} + O(r^{-N}),\ r \to \infty. \tag{3.28}$$

Evaluating the surface integral in (3.27) using (3.28), we obtain

$$E = \pi(N-1)\rho\mathbf{A}\cdot\mathbf{U} - \frac{1}{2}\rho J(B) U^2, \tag{3.29}$$

where $J(B)$ is the content of the body, usually taken as constant. From (3.22) we may set$\mathbf{A} = \mathbf{U}\cdot\mathbf{m}$ for some tensor $\mathbf{m}$ depending oly upon the shape fo the body. Thus

$$E = \frac{1}{2}M_{ij}U_iU_j, M_{ij} = 2\pi(N-1)m_{ij} - \rho J(B)\delta_{ij}. \tag{3.30}$$

We refer to $|Mv$ as the *apparent mass tensor* of the body.

The above analysis of the apparent mass of solids may be found in the textbook of Landau and Lifshitz [**?**]. It is an interesting example of using to advantage at distant boundary for the purpose of computing a property of a finite body, since then only the far field expantion of the potential is needed. It also points directly to the computation of apparent mass by asymptotic analysis.

A *local* determination of **M** can also sometimes be useful (see the next subsection):

$$M_{ij} = -\rho \int_S \Phi_i n_j dS. \tag{3.31}$$

It can be shown from this expresion that M_{ij}, and therefore also m_{ij} is symmetric. One would expect this, but it is not obvious from (3.31). For the proof see exercise 3.4.

Differentiating E and using (3.30) and the symmetry of **M**, we have

$$\frac{dE}{dt} = M_{ij}\dot{U}_i U_j = \mathbf{F} \times \mathbf{U} = \frac{d\mathbf{P}}{dt} \cdot \mathbf{U}, \tag{3.32}$$

providing the shape of the body is independent of time. By Newton's laws $|Pv$ is the total linear momentum of the fluid, and it follows that

$$\mathbf{P} = \mathbf{M} \cdot \mathbf{U}. \tag{3.33}$$

3.1.5 Locomotion by recoil and squirming

We now consider how these results might change if the body is deformable. We assume always that S is impermeable, but that the content $J(B)$ is constant. Internal structures within the body are assumed to deform it in a given way, independent of the resulting inertial forces.

When such an object is placed in an inviscid fluid, will it locomote? Some affirmative answers were given by Saffman [**?**], one of which is easily visualized and analyzed. Consider two dimensisons and an elliptical body of variable eccentricity but fixed area. The body contains a concnetrated mass which can be moved back and forth along a line which will be parallel to the direction of locomotion. As we indicate in Figure 3.1,

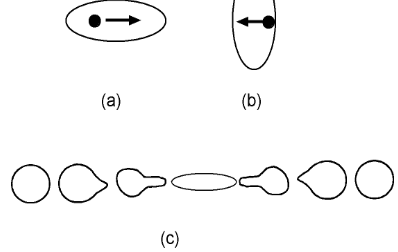


Figure 3.1: (a),(b) depict recoil locomotion in an inviscid fluid. In (a) the mass shifts to the right and body moves left a distance L_1. In (b) the mass shifts an equal distance back to the left, and body moves right a distance $L_2 < L_1$. Over one cycle, the body moves $L_1 - L_2$ to the left. (c) Squirming by a deformable body of constant content.

If the motion starts from rest, conservation of momentum implies that the total momentum of body and fluid will remain zero. We write this as

$$m(U(t) + \Delta U(t)) + P(t) = 0, \tag{3.34}$$

where P is the fluid momentum, is the constant mass of of the body B, U is the velocity of the centroid of the body, and ΔU is the velocity of the center of mass relative to the centroid. We further decompose P:

$$P = M(t)U(t) + P_D(t), \tag{3.35}$$

where $M(t)$ is the apoparant mass of the body at time t, and P_D is the momentum generated by deformation of the surface of the body at time t. The contributions to momentum are therefore four in number: the momentum of the fluid due to motion of the current body shape, the momentum of fluid created by the current surface deformation, the motion of the center of mass of the instantaneous body configuration, and the momentum generated by deformations of the body mass. In general surface deformation can be expected to be acompanied by movement of the center of mass relative to the centroid.

To compute P, we compute the pressure force in the stationary frame. To avoid convergence difficulties, introduce a distant fixed surface Σ enclosing the body surface S and thereby defining a bounded exterior volume V, and a potential ϕ_Σ equal to ϕ on S and zero on Σ. Clearly $\phi_\Sigma \to \phi$ as $\Sigma \to \infty$ since $\phi = O(r^{1-N})$. With $\mathbf{n}$ the outward normal on both surfaces, we then have

$$\begin{aligned}
\mathbf{F}_\Sigma &= \int_S p_\Sigma \mathbf{n} dS \quad \text{(definition of pressure force)}, \\
&= -\rho \int_S \left[\frac{\partial \phi_\Sigma}{\partial t} + \frac{1}{2}(\nabla\phi_\Sigma)^2\right]\mathbf{n} dS, \quad \text{(unsteady Bernoulli theorem)}, \\
&= \int_V \left[\frac{\partial}{\partial t}\nabla\phi_\Sigma + \nabla\cdot(\nabla\phi_\Sigma\nabla\phi_\Sigma)\right] dV - \frac{\rho}{2}\int_\Sigma (\nabla\phi_\Sigma)^2 \mathbf{n} d\Sigma, \\
&= \rho \int_V \frac{\partial}{\partial t}\nabla\phi_\Sigma - \rho\int_S \nabla\phi_\Sigma(\phi_\Sigma \cdot \mathbf{n}) dS \\
&+ \frac{\rho}{2}\int_\Sigma \left[\nabla\phi_\Sigma(\nabla\phi_\Sigma\cdot\mathbf{n}) - \frac{1}{2}(\nabla\phi_\Sigma)^2\mathbf{n}\right] d\Sigma,
\end{aligned} \tag{3.36}$$

Now the body is moving with the fluid relative to the stationary observer, so the first two terms in (3.36) equal the time derivative of the intergral over the moving surface (coonvective derivative), while the remaining terms become negligible as $\Sigma \to \infty$. therefore we have

$$\mathbf{F}_\Sigma = \rho \frac{\partial}{t} \int_V \nabla phi_\Sigma dV + \Sigma \text{ boundary terms}$$

$$= -\rho \frac{\partial}{t} \int_S \phi_\Sigma \mathbf{n} dS. \quad (3.37)$$

Thus, as $\Sigma \to \infty$,

$$\mathbf{F}_\Sigma \to \mathbf{F} = -\rho \frac{\partial}{t} \int_S \phi \mathbf{n} dS. \quad (3.38)$$

Therefore

$$\mathbf{P} = -\rho \int_S \phi \mathbf{n} dS. \quad (3.39)$$

Returning to the present case of motion along a line,

$$P_D = -\rho \int_S \phi_D \mathbf{n} \cdot \mathbf{i}. \quad (3.40)$$

For a homogeneous body dP_d/dt is the force that must be applied to the body to keep its centroid stationary. This averages to zero, as do forces associated with the change of mass distribution within an inhomogeoeous body. We can therefore conclude that *in irrotational inviscid locomotion of a neutrally buoyant body undergoing periodic body formation, not net force is applied to the body.* No mean *thrust* can be generated which could accelerate the body. This kind of locomotion has some features in common with Stokesian swimming, since each cycle of motion can result in a net displacement, and is quite different from Eulerian modes ivolving vortex forces.

If there is no body deformation, but only shifting of internal mass, (3.34) and (3.35) combine to give

$$\langle U \rangle = -\Big\langle \frac{m \Delta U}{m + M} \Big\rangle. \quad (3.41)$$

If the body is homogeneous but the surface deforms, we have similarly

$$\langle U \rangle = -\Big\langle \frac{P_D}{m + M} \Big\rangle. \quad (3.42)$$

In both cases Saffman [**?**] gives examples of locomotion, of the kind shown in Figure 3.1.

Locomotion by recoil and squirming are thus examples of mechanisms relying solely on inertial forces realized with no vorticity in the fluid. Whereas all Eulerian locomoters utilize the fluid inertia, the vortex forces tend to dominate over any of these irrotaional inertial forces. The vortex forces lead an invisicid *thrust*, which in steady locomotion is balanced by viscous drag. One questiion of interest is whether or not recoil swimming can be realized in a fluid of small but finite viscosity. Squirming, on the other hand, since it involves deformatiions which are not reciprocal, could succeed at all Reynolds numbers, see also Exercise 3.6.

3.2 The swimming of slender fish

We now take the mechanical principles underlying Eulerian swimming of a thin, flexible creature, with the primary aim of understanding how the locomotion and maneuvering of fish. For summaries of the related biology, see [**?**], chapter 2, and [**?**].

3.2.1 Small-perturbation theory

Although there are certainly exceptions, many fish change shape rather gradually along the anterior-posterior axis. It is there for natural to begin a study of fish swimming by considering a slender, neutrally buoyant organism.

The implications of "slender" need to be carefully considered. The basic idea is that the velocity perturbations of the water caused by slender fish in steady locomotion at speed U should be small compared to U. It is necessary that the cross-sectional area of the fish should change gradually along its length. It is sufficient that the body be smooth and that any plane tangent to the body should make a small angle with the line of swimming. An ideal candidate of this kind would then b an eel-like creature whose body cross-section is circular with a radius which changes gradually and is zero at both extremities. Unfortunately, it turns out that if these conditions are met the theory we are about to describe would predict that now swimming is possible, in the sense that small undulatory moments do not result in any mean thrust. So it will be necessary to widen the acceptable "slender" bodies to include those which change abruptly at the downstream tail fin (the caudal fin). Also surface slopes need not be small at the nose of the fish, provided that large angles occur over a small fraction of the length. However slenderness can be violated at fins where the *upstream* edge angle is not small.

We shall restrict our study to a body with the following properties: (1) When "stretched straight" it is laterally symmetric about the mid-body y-z plane, see Figure 3.2. This is a property of most fish. (2) With the exception of the vicinity of the nose and the downstream vertical edge of the caudal fin, the body is smooth and surface slopes are small. (The addition of mid-body fins is considered in subsection ??.) (3) The cross-sectional area is zero at both ends, the downstream section being the edge of the caudal fin, hence a line segment, and the upstream section reducing to a point.

See figure on page 99 of MS&F.

Figure 3.2: Notation for slender fish. (a) “Reality.” (b) Slender fish without mid-body fins. $|s'(x)| \ll 1, |\partial h/\partial x| \ll 1, |\partial h/\partial t| \ll U$.

Movements are assumed to be lateral (in the z-direction), and for a small-amplitude theory we require

$$|\partial h/\partial x| \ll 1, |\partial h/\partial t| \ll U, \tag{3.43}$$

where U is the swimming speed in the direction of negative x, a quantity that will be take as a constant. These assumptions insure that the fluid velocity observed in the moving frame will differ from U by only a small amount. Thus we may make the approximation

$$\frac{d}{dt} \approx \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} \tag{3.44}$$

for the material derivative. From the dynamical viewpoint, (3.44) carries with it the implication that a *stationary* observer can select a thing *water slice* between to nearby planes $x =$ constant, and observed the swimming fish pass through the slice with significantly deforming the two bounding surfaces. At the same time the intersection of the slice with the body of the fish will present an essentially two-dimensional time-dependent motion in the plane of the slice, caused by the spatial and temporal undulations, as well as the variation in the shape of the cross-section, along the length of the fish. This problem is similar to that of two-dimensional flow past a moving solid ellipse (the elliptical cross section being a reasonable first approximation for many fish). There is one new feature– the *area* of the cross-section can change, introducing an effective source flow. However by the assumed lateral symmetry this will not produce z-momentum and need note therefore be considered in the lateral momentum balance. In fact for the present discussion it is sufficient to take the fish to be arbitrarily thin in the $z-$ direction, a *flat fish*, with a slender contour save for the downstream fin edge.

We turn now to Lighthill's pioneering analysis of the foregoing problem. His small-amplitude theory is based upon an insightful division of the calculation into two different evaluations of the same quantity, namely the rate of working of the fish's body on the fluid. First, this quantity is calculated directly, utilizing only the definitions of apparent mass and of rate of working. Then, the law of conservation of energy is used to relate this rate of working to the whatever work is done by thrust (as yet unknown!) and the creation of kinetic energy in the fluid. This brings the thrust T into the picture, all other quantities being directly computable from the motion of the body. One then solves and averages to obtain a mean thrust.

First then, we compute directly the rate of working of the inertial forces. The velocity of the cross section in a water slice must take into account both the instantaneous body movements and the body at that section, as well as the apparent motion from the passage of the slice down a wavy body. An observer on the water slice (moving with velocity U) thus sees the section velocity

$$w = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} \equiv Dh, \tag{3.45}$$

in terms of the lateral displacement $h(x,t)$ of the mid-body surface. If $m(x)$ is the apparent mass of the section a distance x from the nose, the lateral force exerted by the body on the water slice is then the material derivative of mw.[1]

$$F_z = D(mw). \tag{3.46}$$

The rate of working done by the fish as it moves sections of its body laterally can now be evaluated as

$$\mathcal{W}(t) = \int_0^L F_z \frac{\partial h}{\partial t} dz = \int_0^L D(mw)\frac{\partial h}{\partial t} dz, \tag{3.47}$$

where L is the length of the fish. Manipulating the integrand, we have

$$\begin{aligned}\mathcal{W}(t) &= \int_0^L D\big(mw\frac{\partial h}{\partial t}\big)dx - \int_0^L mw\frac{\partial}{\partial t}(Dh)dx \\ &= \frac{\partial}{\partial t}\int_0^L \big(mw\frac{\partial h}{\partial t} - \frac{1}{2}mw^2\big)dx + \big[Umw\frac{\partial h}{\partial t}\big]_{x=L}.\end{aligned} \tag{3.48}$$

[1] For simplicity we take the apparent mass of all sections to be independent of time, even though it is known that some fish can vary at will the depth of the caudal fin.

Here we have used $m(0) = 0$, i.e. the assumption that the nose of the fish have zero apparent mass in lateral motion.

We see from (3.48) that the *mean* rate of working of the fish depends only upon the conditions at the posterior or downstream end:

$$\langle \mathcal{W} \rangle = U \langle m(L) w(L,t) \frac{\partial h}{\partial t}(L,t) \rangle. \tag{3.49}$$

This is already an interesting outcome, since it is very different from flagellar propulsion, where all sections of the body contribute to thrust. Also it is intriguing biologically. The almost universal occurrence among fish of a well-developed caudal fin can be taken as evidence of its importance to propulsion.

The second stage of the argument brings in the conservation of energy. The system we consider consists of the body surface and the water slices it intersects.[2] We suppose that for whatever reason the interaction of the waving body and the water has resulted in a net pressure thrust T. If not kinetic energy were being created in the water slices we would have $W = UT$ within the small-amplitude theory– the rate of working of the fish would be fully realized as thrust. However, in fact the rate of creation of kinetic energy at each water slice is, from out general results on apparent mass, $D(\frac{1}{2}mw^2)$. Thus conservation of energy demands

$$\mathcal{W}(t) = UT(t) + \int_0^L D(\frac{1}{2}mw^2)dx$$

$$= UT + \frac{\partial}{\partial t}\int_0^L \frac{1}{2}mw^2 + \frac{1}{2}\big[Umw^2\big]_{x=L}. \tag{3.50}$$

The second term on the right of (3.50) is instantaneous rate of change of the kinetic energy in the system, while the third accounts for the kinetic energy which is shed into the wake at the downstream edge of the caudal fin (see the subsection to follow).

We now compare the two expressions we have for $\mathcal{W}$, namely (3.48) and (3.50), and use $w = Dh$ to obtain

$$T = m(L)[wW - \frac{1}{2}w^2]_{x=L} - \frac{\partial}{\partial t}\int_0^L mw\frac{\partial h}{\partial x}dx, \tag{3.51}$$

where $W = \frac{\partial h}{\partial t}(L,t)$. We thus see that the mean thrust is given by

$$\langle T \rangle = m(L)\langle [wW - \frac{1}{2}w^2]_{x=L}\rangle, \tag{3.52}$$

[2]Obviously other factors– the mass of the body, the energy expended in the muscles– could be taken into account in an enlarged thermodynamic system. Here we consider only the water, its energy and the forces doing work on it.

and so is fully determined entirely by conditions at the edge of the caudal fin. This is again a somewhat surprising and informative result, since the thrust is actually realized by adding pressure forces over the entire body. An alternative expression for thrust is

$$\langle T\rangle = \frac{m(L)}{2}\langle h_t^2 - U^2 h_x^2\rangle \tag{3.53}$$

An appropriate efficiency of swimming is the *Froude efficiency*

$$\eta = \frac{U\langle T\rangle}{\langle \mathcal{W}\rangle}. \tag{3.54}$$

From (3.48) and (3.52) we find

$$\eta = 1 - \frac{1}{2}\frac{\langle w^2\rangle}{\langle Ww\rangle} = \frac{1}{2}\frac{\langle h_t^2 - U^2 h_x^2\rangle}{\langle h_t^2 + U h_t h_x\rangle}. \tag{3.55}$$

Comparing (3.52)and (3.55) we see that in order to maintain positive thrust and reach high efficiency simultaneously, w and W should be positively correlated while w should be kept as small as possible. If we adopt a progressive wave motion for the lateral excursions of the body,

$$h(x,t) = h_0 \sin(kx - \omega t), \tag{3.56}$$

we see that

$$W = -h_0\omega \cos(kx - \omega t),\ W = -h_0(\omega - Uk)\cos(kx - \omega t), \tag{3.57}$$

and

$$\langle T\rangle = \frac{m(L)h_0^2}{4}(\omega^2 - U^2k^2),\ \mathcal{W} = \frac{kUm(L)h_0^2}{2}(V - U),\ V = \omega/k, \tag{3.58}$$

and so

$$\eta = \frac{U+V}{2V},\ V = \omega/k. \tag{3.59}$$

We see that V must exceed U, that is, the swimming speed (motion to the left) cannot exceed the wave speed (motion to the right), and that efficiency is a maximum at just that point, $U = V$, where thrust vanishes. The positive correlation of w and W implies that fin slope $h_x(L,t)$ and $-h_t$ reach maxima and minima simultaneously, see Figure 3.3.

If $h_0 = \epsilon L$ is an amplitude of lateral movement, and if a typical frequency of movement is U/L, the mean thrust predicted by the small-amplitude theory is $O(m(L)U^2\epsilon^2$, an so is second order in the lateral amplitude. o understand the source of this small thrust is challenging, since the system seems to be utilizing the *inertial* forces of apparent mass. The body pushes against the water slice as it flows past. No net work would be done by a body which continued indefinitely, but here the water slice gets to the tip of the caudal fin and flows into the wake. The disturbance caused in the wake is indicative of vorticity that is shed at the caudal fin, see subsection ??. This then is a hint that the source of the thrust must be understood in terms of small vortex forces in the vicinity of the caudal fin. The mechanism will be discussed below, see ??. The vortical wake downstream of a fish can be visualized and when seen from above consists of a series of vortices o alternating orientation, see Figure 3.4 below.

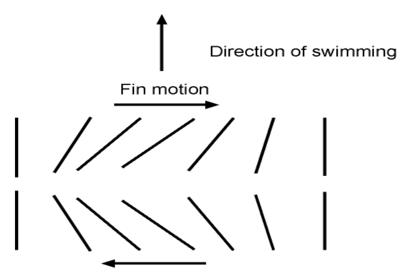


Figure 3.3: Fin slope and speed reach a maximum simultaneously.

3.2.2 Finite amplitude theory

Although the assumption of the slenderness is a natural one for the analysis of fish locomotion, the assumption of small perturbations is an ad hoc simplification which makes the problem linear. It should be possible to exploit the slenderness of the body in a nonlinear theory, provided the geometry is allowed to depart substantially from the stretch-straight position.

This extension was realized by Lighthill, see [**?**] chapter 5, through an argument based entirely upon the momentum balance. One reason for giving up the energy arguments is that at finite amplitudes it is no longer clear how to divide up the rate of working between the lateral inertial forces and the thrust. In the linear theory, the introduction of U at the outset provided a framework for the deduction of the work done by thrust, but this is possible because of the small amplitude setting.

In the derivation we give here, we shall modify Lighthill's argument by bringing in some aspects of the energy balance, as it pertains to the wake in particular. let the arc length s be measured along the curved surface lateral symmetry from the nose to the caudal fin edge $s = L$, and let $x(s,t), z(s,t)$ denote the intersection of this surface of symmetry with the $x - z$ plane, observed in the stationary frame. The tangent along the axis of the fish and normal vector to the plane of symmetry are then (x_s, z_s) and $(-z_s, x_s)$ respectively. Set $u = \mathbf{u} \cdot \mathbf{t}, w = \mathbf{u} \cdot \mathbf{n}$, where $\mathbf{u} = (x_t, z_t)$ is the velocity of a given section of the fish in the $x - z$ plane.

The momentum carried by a given section is then $m(s)w(s,t)\mathbf{n}(s,t)$, where again m denoted the apparent mass of the section, assumed independent of time. Summed up along the length of the fish, the rate of change in time of this total momentum must equal the momentum lost downstream, plus the force $\mathbf{F}$ exerted by the fluid on the fish, minus any pressure force $f_\Sigma \mathbf{t}$ exerted by the fluid on the plane Σ of Figure 3.4:

$$\frac{d}{dt}\int_0^L mw\mathbf{n}ds = -[umw\mathbf{n}]_{s=L} - f_\Sigma \mathbf{t} + \mathbf{F}. \tag{3.60}$$

We point out that we have here neglected the tangential momentum flux $[u^2\mathbf{t}]_{ss=L}$ through Σ. This is because of slenderness. A slender body waving laterally at large amplitude cannot give rise to large tangential velocity. In the linear theory, u is of order ϵ^2 whereas w is of order ϵ. In the finite amplitude theory w is allowed to be $O(1)$, but u remains $O(\epsilon^2)$.

See figure on page 106 of MS&F.

Figure 3.4: Notation for finite amplitude theory. The plane Σ is spanned by the vectors $\mathbf{n}(L,t)$ and $\mathbf{t}(L,t) \times \mathbf{n}(L,t)$.

To compute f^Σ, Lighthill uses an argument based entirely upon the momentum balance through the unsteady Bernoulli theorem. We shall obtain his result by a different method, reverting to the energy balance in fluid, This approach has the advantage of dealing at all times with absolutely convergent integrals, some of which are negligible by slenderness. Let E_w be the total energy in the wake of the fish, viewed as possibly large bout bounded for motion started from rest. Consider the now the energy balance of the stationary observer:

$$\dot{E}_w = W_{S_w} + W_\Sigma, \tag{3.61}$$

where W_{S_w} and W_Σ are the rates of working of the two indicated surfaces. Now it is clear the W_{S_w} must vanish, since S_w is a surface of fluid particles passing over the downstream edge of the caudal fin. The surface is also a *vortex sheet* (see the following subsection), carrying the vorticity shed by the caudal fin, but it is nevertheless simply a material surface, across which the pressure is continuous. Hence it can do no work.

The left-hand side of (3.61) may be evaluated using conservation in a domain with a moving boundary:

$$\dot{E}_w = \rho \int_{V_w} \nabla\phi \cdot \nabla\phi_t dV + \frac{\rho}{2}\int_\Sigma (\nabla\phi)^2 u_\Sigma d\Sigma, \tag{3.62}$$

where u_Σ is the velocity component normal to Σ. At this point we bring in the unsteady Bernoulli formula to obtain

$$\dot{E}_w = \rho \int_\Sigma [p/\rho + \frac{1}{2}\nabla\phi^2]\mathbf{t}\cdot\nabla\phi d\Sigma + \frac{\rho}{2}\int_\Sigma (\nabla\phi)^2 u_\Sigma d\Sigma$$

$$= +\frac{\rho}{2}\int_\Sigma (\nabla\phi)^2 u_\Sigma d\Sigma \tag{3.63}$$

$$\approx \frac{1}{2}[mw^2u]_{s=L}. \tag{3.64}$$

Notice what is happening here. (3.63) follows from the smallness of $\mathbf{t}\cdot\nabla\phi$ in slebder-body theory, whereas (3.64) uses the fact that the only contribution is from the translational part of the rigid body motion of Σ, normal to itself with *spatially constant* velocity equal to $u(L,t)$. The effect of rotation does not contribute because the integral is an absolutely convergent integral of a product of a function odd (the rotational part of u_Σ)and even ($(\nabla\phi)^2 \sim O(r^{-4})$ in local 2D polar coordinates. We have also related the kinetic energy in Σ to the apparent mass at $s = L$. Note that (3.64) tells us that wake energy is new exclisively to the shedding of vorticity from the caudal fin.

Finally, we consider w_Σ. This term must be a linear combination of the form

$$W_\Sigma = Au(L,t) + B\Omega(L,t), \tag{3.65}$$

where Ω is the instantanwous angular speed of Σ. Hoever, since (3.64) is independent of Ω, involving only $u(L,t)$, we conclude that $B = 0$ and

$$W_\Sigma = -f_\Sigma u(L,t) = -\frac{1}{2}[mw^2u]_{s=L}. \tag{3.66}$$

Thus,

$$f_\Sigma = -\frac{1}{2}[mw^2]_{s=L}. \tag{3.67}$$

Returning to (3.60), we now have

$$\mathbf{F} = [umw\mathbf{n} - \frac{1}{2}mw^2\mathbf{t}]_{s=L} + \frac{d}{dt}\int_0^L mw\mathbf{n}ds. \tag{3.68}$$

Using $u\mathbf{n} - w\mathbf{t} = \mathbf{u} \times \mathbf{j}$, (3.68) yields Lighthill's equation:

$$(F_x, F_z) = [-mw(z_t, -x_t) + \frac{m}{2}w^2(x_s, z_s)]_{s=L} + \frac{d}{dt}\int_0^L mw(-z_s, x_s)ds. \tag{3.69}$$

This elegant result, which provides an expression for side force as well as thrust, can be used to study such large-amplitude swimming maneuvers as turning and starting, where the lateral velocity of the fin is comparable to the swimming velocity.

The average thrust deveoped by the fish is then

$$\langle T\rangle = -\langle F_x\rangle = \langle [mwz_t - \frac{m}{2}w^2x_s]_{s=L}\rangle. \tag{3.70}$$

As an application we consider a progressive wave of sinusoidal form. Let the wave number k be unity by the choice of unit of length. The wave velocity V will be to the right and swimming with speed U to the left. An observer moving with the crests of the wave will thus see, assuming an inextensible body, sections moving with velocity $-Q\mathbf{t}$ where $Q = \alpha V$. (We use the same notation here as for the flagellum in the Gray-Hancock analysis of Stokesian resistive-force propulsion, except for a sign change in the definition of U.) Here αL is the length of the fish projected only the $x-$ axis. We set $z = h_0 \sin(s - Qt)$. The stationary observer sees a section velocity $\mathbf{u} = (V - U)\mathbf{i} - Q\mathbf{t}$, and $w = \mathbf{u} \cdot \mathbf{n},\ \mathbf{n} = (-z_s, x_s) = -z_s(V - U)$. Since $z_s z_t = -h_0^2 Q \cos^2(s - Qt)$ and $x_s = \alpha$ we obtain

$$\langle T\rangle = \frac{m(L)h_0^2}{2\alpha}\big[V(V-U) - \frac{\alpha^2}{2}(V-U)^2\big]. \tag{3.71}$$

This is seen to reduce to the linear result $\frac{m(L)h_0^2}{2\alpha}(V^2 - U^2)$ when $\alpha = 1$.

The Froude efficiency is now defined defined by UT divided by the total effort, the latter being equal to UT plus the rate of energy loss into the wake:

$$\eta = \frac{UT}{UT - \frac{1}{2}\langle [mw^2u]_{s=L}\rangle}. \tag{3.72}$$

From this we obtain

$$\eta = 1 - \frac{\alpha^2}{2}\big(\frac{U}{V} - 1\big), \tag{3.73}$$

which is large than the linear result but reduces to it when $\alpha = 1$.

3.2.3 Vortex shedding

The vorticity shed from the caudal fin into the wake is depicted in Figure ??. Note that since the fin at $s = L$ reduces in section to a line, the vorticity is shed into a sheet which is a material surface (the surface S_w of Figure ??). We now want to obtain expressions for the rate of injection of vorticity into the wake in terms of conditions at the edge of the caudal fin.

To do this we shall make use of Kelvin's theorem. Consider the situation shown in Figure 3.5. The points A, B are to viewed as infinitesimally close, on either side of the caudal fin edge, immediately downstream of the edge. Now in general the velocity $\mathbf{u}_A$ on the side of the fin which corresponds to point A, evacuated at the edge, is not equal to the corresponding velocity on the opposite face, $\mathbf{u}_B$. It is this discontinuity of velocity across the fin edge that is maintained into the wake, producing a vortex sheet.

See figure on page 99 of MS&F.

Figure 3.5: Vortex shedding from the caudal fin. At the instant depicted, streamwise vorticity is being shed.

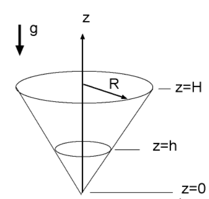
A short time Δt later, the points A, B will have moved to A', B', still infinitesimally close to the sheet, but separated in position on the sheet because of the discontinuity in velocity. Assuming that A, B were separated by a lateral distance ϵ, a line segment joining A' to B' will have the property that an observer on this line segment moving from A' to B' will see $\mathbf{u}_A$ as the local velocity on the first one-half of the length of the segment, and $\mathbf{u}_B$ on the remaining one-half of the segment.

3.3 Exercises

3.1. An Euler flow with no vortex force is called a *Beltrami field*. Show that $\mathbf{u} = \nabla \times \nabla \times \mathbf{A} + \alpha \nabla \times \mathbf{A}$ is a Beltrami field for any $\mathbf{A}$ satisfying $\nabla^2 \mathbf{A} + \alpha^2 \mathbf{A} = 0$.

3.2 Water fills the right circular cone shown in the figure, gravity acting down. At time $t = 0$, the bottom section at $z = h$ is sliced off, so that water flows out the bottom. At time $t = 0+$ however, the water has not moved, but the pressure at *both* the bottom section $z = h$ and the top section $z = H$ is the ambient pressure p_0. The question is, how would you determined the pressure distribution on the inner surface of the cone, at time $t = 0+$? Notice that if $h = 0$ the pressure distribution is that of the static system, $p = p_0 + \rho g(H - z)$, where $\rho =$ the density of water. Also, if h is close to H, the pressure is just p_0 throughout.

Make use of the unsteady Bernoulli theorem, assuming the developing flow is potential. Expand the potential as a Taylor series in time.You should formulate a problem to be solved for ϕ, but you do not have to solve this problem or to find the pressure distribution explicitly.



3.3. The potential flow of a uniform stream $\mathbf{U} = (U(t), 0, 0)$ around a fixed sphere $r = a$ has the potential

$$\phi = U(t)\big(x + \frac{a^3 x}{2r^3}\big).$$

What is the potential of a sphere moving through fluid otherwise at rest with speed $(U(t), 0, 0)$? Evaluate the pressure using the unsteady Bernoulli theorem, compute the pressure force on the sphere, and compare your result against our general expression for apparent mass.

3.4. Show that the apparent mass tensor $\mathbf{M}$ is symmetric, $M_{ij} = M_{ji}$. Do this by considering $\int (\Phi_j n_i - \Phi_i n_j) dS$, using the boundary condition on S to write the integral in the form

$$\int \big(\Phi_j \frac{\partial \Phi_i}{\partial x_k} - \Phi_i \frac{\partial \Phi_j}{\partial x_k}\big) n_k dS.$$

Then apply green's theorem to the external domain with $\Phi = O(r^{(1-N)})$ at infinity.

3.5. Using (3.25) to define momentum, and assuming (3.28), show that an erroneous value of apparent mass is obtained by taking Σ to be spherical. (Hint: Write the integral in the form $\rho \int_V \nabla \cdot (\mathbf{r} \cdot \nabla \phi) dV$.)

3.6. Show that a homogeneous, neutrally buoyant squirmer, starting from rest with a time-reversible boundary motion, cannot swim in an inviscid fluid. (Suggested by Charles Peskin.)