Chapter 8

The boundary layer

The concept of the boundary layer is a classic example of an applied science greatly influencing the development of mathematical methods of wide applicability. The key idea was introduced in a 10 minute address in 1904 by Ludwig Prandtl, then a 29 year old professor in Hanover, Germany. Prandtl had done experiments in the flow of water over bodies, and sought to understand the effect of the small viscosity on the flow. Realizing that the no-slip condition had to apply at the surface of the body, his observations led him to the conclusion that the flow was brought to rest in a thin layer adjacent to the rigid surface. His reasoning suggested that the Navier-Stokes equations should have a somewhat simpler form owing to the thinness of this layer. This led to the equations of the viscous boundary layer. Boundary-layer methods now occupy a fundamental place in many asymptotic problems for partial differential equations.

8.1 The limit of large Re

Let us consider the steady viscous two-dimensional flow over a flat plate aligned with a uniform stream $(U, 0)$. In dimensionless variables the steady Navier-

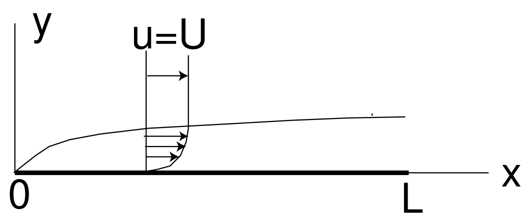

Figure 8.1: Boundary layer on a finite flat plate.

Stokes equations in two dimensions may be written

$$\mathbf{u} \cdot \nabla u + p_x - \frac{1}{Re}\nabla^2 u = 0, \tag{8.1}$$

$$\mathbf{u} \cdot \nabla v + p_y - \frac{1}{Re}\nabla^2 v = 0, \tag{8.2}$$

$$u_x + v_y = 0. \tag{8.3}$$

We are dealing with the geometry of figure 8.1. The boundary layer is seen to grow in thickness as x moves from 0 to L. This suggests that the term $\mathbf{u} \cdot \nabla u$ in (8.1) has been properly estimated as of order U^2/L in the dimensionless formulation, and so should be taken as $O(1)$ at large Re in (8.1). If this term is to balance the viscous stress term, then the natural choice, since the boundary layer on the plate is observed to be so thin, is to assume that the y-derivatives of u are so large that the balance is with $\frac{1}{Re}u_{yy}$. Thus it makes sense to define an *stretched* variable $\bar{y} = \sqrt{Re}y$. If we now apply the stretched variable to (8.3), still taking u_x as of order unity, then in order to keep this essential equation intact we must compensate the stretched variable $\bar{y}$ by a streched form of the y-velocity component:

$$\bar{v} = \sqrt{Re}v. \tag{8.4}$$

Prandtl would have been comfortable with this last definition. The boundary layer on the plate was so thin that there could have been only a small velocity component normal to its surface. Thus the continuity equation will survive our limit $Re \to \infty$:

$$u_x + \bar{v}_{\bar{y}} = 0. \tag{8.5}$$

Returning now to consideration of (8.1), retain the pressure term p_x as $O(1)$ as well so that the simplified equation, obtained in the limit $Re \to \infty$ in the stretched variables, amounts to dropping the term $\frac{1}{Re}u_{xx}$:

$$uu_x + \bar{v}u_{\bar{y}} + p_x - u_{\bar{y}\bar{y}} = 0. \tag{8.6}$$

Finally, using these stretched variables in (8.2) we have

$$p_{\bar{y}} = -\frac{1}{Re}(u\bar{v}_x + \bar{v}\bar{v}_{\bar{y}} - \bar{v}_{\bar{y}\bar{y}}) + \frac{1}{Re^2}\bar{v}_{xx}. \tag{8.7}$$

Thus in the limit $Re \to \infty$ the vertical momentum equation reduces to

$$p_{\bar{y}} = 0. \tag{8.8}$$

We thus see from (8.8) that the pressure does not change as we move vertically through the thin boundary layer. That is, the pressure throughout the boundary layer at a station x must be the pressure outside the layer. At this point a crucial contact is made with inviscid fluid theory. The "pressure outside the boundary layer" should be determined by the inviscid theory, since the boundary layer is thin and will presumably not disturb the inviscid flow very

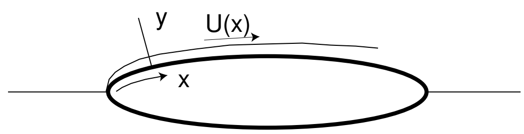

Figure 8.2: Boundary layer over a general body with varying $U(x)$.

much. In particular for a flat plate the Euler flow is the uniform stream- the plate has no effect- and so the pressure has its constant free-stream value.

Prandtl's striking insight is clearer when we consider flow past a general smooth body, as in figure 8.2. Since the boundary layer is again taken as thin in the neighborhood of the body, curvilinear coordinates may be introduced, with x the arc length along curves paralleling the body surface and y the coordinate normal to these curves. In the stretched variables, and in the limit for large Re, it turns out that we again get (8.6)-(8.8), only now (8.8) must be interpreted to mean that the pressure is what would be computed from the inviscid flow past the body. If p_0, U_0 are the free stream values of p, u, then Bernoulli's theorem for steady flow yields along the body surface

$$p_{euler} = p_0 + \frac{1}{2} - \frac{1}{2}U^2(x) = p(x), \tag{8.9}$$

and it is this $p(x)$ which now applies in the boundary layer, by (8.8). Thus the inviscid flow past the body determines the pressure variation which is then impose on the boundary layer through the now known function p_x in (8.6).

We note that the system (8.6)=(8.8) are usually called the *Prandtl boundary-layer equations*.

We are giving here the essence of Prandtl's idea without any indication of possible problems in implementing it for an arbitrary body. The main problem which will arise is that of *boundary layer separation*. It turns out that the function $p(x)$, which is determined by the inviscid flow, may lead to a boundary layer which cannot be continued indefinitely along the surface of the body. What can and does occur is a breaking away of the boundary layer from the surface, the ejection of vorticity into the free stream, and the creation of free separation streamline similar to the free streamline of the Kirchoff flow we considered in chapter 6. Separation is part of the stalling of an airfoil at high angles of attack, for example.

8.2 Blasius' solution for a semi-infinite flat plate

We now give the famous Blasius solution of the boundary layer past a semi-infinite flat plate; geometrically the problem is that of figure 8.1 with $L = \infty$. The fact that the plate is infnite will mean that the boundary layer extends to infinity. We will comment on this later. For the moment simply note that we have now expelled the length L from the problem, even though we used it previously to define a Reynolds number, which number we then let tend to infinity. Without a length in the problem, however, it becomes much simpler to solve, because the no-slip conditions applies on the entive line $x > 0, \bar{y} = 0$.

We recall that for the aligned flat plate the pressure in Prandtl's boundary layer equations is zero, so we seek to solve

$$uu_x + \bar{v}u_{\bar{y}} - u_{\bar{y}\bar{y}} = 0, \quad u_x + \bar{v}_{\bar{y}} = 0, \tag{8.10}$$

subject to the conditions $u = \bar{v} = 0, \bar{y} = 0, x > 0$, and $u \to 1$ as $\bar{y} \to \infty, x > 0$. We can satisfy the solenoidal condition in the usual way with a boundary-layer stream function $\bar{\psi} = \sqrt{Re}\psi$ such that $u = \bar{\psi}_{\bar{y}}, \bar{v} = -\bar{\psi}_x$. We then observe that our problem has a *self-similar* structure in the following sense. The equations and conditions are invariant under the group of "stretching" transformations

$$x \to Ax, \bar{y} \to B\bar{y}, \bar{\psi} \to C\bar{\psi}, \tag{8.11}$$

provided that $A = B^2$ and $B = C$. Indeed, the condition $u = 1$ transforms to $uC/B = 1$ so we must have $B = C$. Also the term uu_x scales like $\frac{C^2}{AB^2}$ while $u_{\bar{y}\bar{y}}$ scales like C/B^3, and the equality of these two factors requires $A = BC$. The remaining terms follow suit and so (8.10) and the conditions are invariant under the stated conditions $A = B^2 = C^2$. Now the combination $\eta = y/\sqrt{x}$ is then invariant under (8.11), and therefore so is the equation $\bar{\psi} = \sqrt{x}F(\eta)$ for any function F. If we assume a $\bar{\psi}$ of this form and substitute it into

$$\bar{\psi}_{\bar{y}}\bar{\psi}_{x\bar{y}} - \bar{\psi}_x\bar{\psi}_{\bar{y}\bar{y}} - \bar{\psi}_{\bar{y}\bar{y}\bar{y}} = 0, \tag{8.12}$$

it is straightforward to show that we get

$$-\frac{1}{x}\left[\frac{1}{2}FF'' + F'''\right] = 0. \tag{8.13}$$

The conditions to be satisfied are then

$$F(0) = F'(0) = 0, \quad F' \to 1,\ \eta \to \infty. \tag{8.14}$$

The simplest way to solve this problem is to replace it by the following initial-value problem:

$$\frac{1}{2}GG'' + G''' = 0, G(0) = G'(0) = 0, G''(0) = 1. \tag{8.15}$$

When this problem is solved (a simple matter using ode45 in MATLAB on an interval $0 < \eta < 5$ say, we obtain values of $G'(\eta)$ similar to figure 8.3 (the solution of the actual problem) but asymptoting to $c = 2.0854$ instead of 1. However if $G(\eta)$ is a solution of our equation, so is $AG(A\eta)$ for any constant A. Since $G \sim c\eta + o(\eta), \eta \to \infty$, we set

$$F(\eta) = c^{-1/2}G(c^{-1/2}\eta). \tag{8.16}$$

This give the curve for $F'(\eta)$ shown in figure 8.3. One finds

$$F(\eta) \sim \eta - 1.7208 + o(1), \quad \eta \to \infty, \tag{8.17}$$

and also $F''(0) = c^{-3/2} = .332$

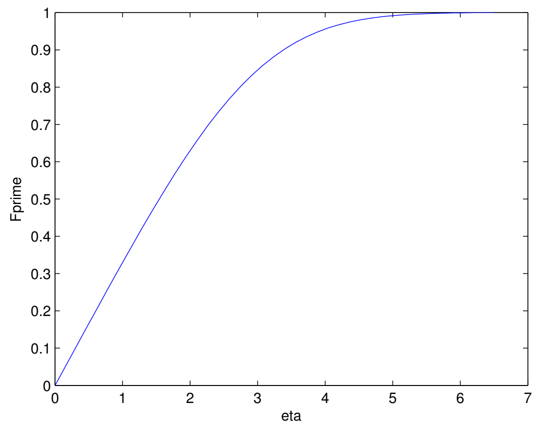

Figure 8.3: The Blasius boundary layer velocity profile.

8.2.1 Discussion of the Blasius solution

Recalling that the dimensional form of the stream function is $UL\psi$, the dimensional form of ψ is $UL\psi = ULR^{-1/2}\bar{\psi}$. In terms of x, y which have dimensions, $\bar{\psi} = (x/L)^{1/2} F\Big(\frac{\sqrt{R}y/L}{\sqrt{(x/L)}}\Big)$. Thus with dimensions fully restored the stream function may be written

$$\sqrt{U\nu x}F\Big(y\sqrt{\frac{U}{\nu x}}\Big). \tag{8.18}$$

confirming the fact that the problem we have solved is free of a length L. From the asymptotic behavior of F for large η we then have the dimensional stream function for large y in the form

$$Uy - 1.7208\sqrt{U\nu x} + o(1), y \to \infty. \tag{8.19}$$

This combination of terms vanishes when $y = 1.7208\sqrt{\nu x/U}$. This shows that well away from the plate the streamlines look like those over a thin parabolic cylinder. This process of "lifting" the distant streamlines makes the plate look like it has some thickness, which grows downstream as $\sqrt{x}$. This thickness, which has been given the term *displacement thickness*, can be understood from the nature of the volume flux in the boundary layer. As the boundary layer grows with increasing x more and more fluid parcels originally moving with the free-stream velocity U, are found to be moving more slowly. This depleted volume flux near the wall, which increases with x, must be compensated by an outward full of volume away from the wall. It is this outward flux which lifts the streamlines to their parabolic form.

The displacement thickness can be given a precise definition as follows. Let

$$\delta(x) = \int_0^\infty (1 - u/U)dy. \tag{8.20}$$

Then $Ud\delta(x) = V(x)dx$ where $V(x)$ is the compensating upward velocity is equal to the integral of $U - u$ through the layer, which is the reduced volume flux through the boundary layer. But according to (8.18) $u = UF'\Big(y\sqrt{\frac{U}{\nu x}}\Big)$ and so, since $F(\eta) \sim \eta - 1.7208 + o(1)$ we have

$$\delta(x) = \sqrt{\frac{\nu x}{U}} \lim_{\eta\to\infty} (\eta - F(\eta)) = 1.7208\sqrt{\frac{\nu x}{U}}. \tag{8.21}$$

Thus

$$V(x) = 1.7208U\delta'(x) = .8604\sqrt{\frac{U\nu}{x}} = dy/dx, \tag{8.22}$$

where $y(x) = 1.7208\sqrt{\nu x/U}$ is the zero streamline of the "effective body" whose thickness we may now identify with the displacement thickness as defined.

It is interesting to ask what error is being made if we substitute the Blasius solution into the full Navier-Stokes equations and look at the remainder. We consider here only the dimensionless form of the x-momentum equation. There the terms we expelled to get the boundary layer equation were $\frac{1}{R}u_{xx}$ and p_x. Substituting $u = F'(\eta)$ we obtain the exact equation

$$p_x - \frac{1}{x}\Big[\frac{1}{2}FF'' + F'''\Big] - \frac{\eta}{4x^2R}\Big[3F'' + \eta F'''\Big] = 0. \tag{8.23}$$

We see that the second bracketed term fails to be smaller that the first when $xR = O(1)$. Thus near the front edge of the plate the boundary layer equations are not uniformly valid. In a small circular domain of order $1/R$ in radius about the origin, the full Navier-Stokes equations can be shown to govern the fluid flow. This small non-uniformity does not affect the validity elsewhere, however. We can assert this because of the existence of the Blasius solution, and the fact that experimental measurements confirm its validity at large R.

As a final remark concerning the Blasius solution, we note that the *finite* flat plate, of length L, can be approached with exactly the same apparatus. Although the Prandtl boundary-layer equations fail to hold at $x = L$ as well as $x = 0$, the development of the layer on the plate is unaltered to first order. In paticular the drag on the plate, accounting fro both sides is given by

$$D = 2\mu \int_0^L u_y(x, 0)dx = 2\mu U \int_0^L \sqrt{\frac{U}{\nu x}}F''(0)dx \tag{8.24}$$

This yields

$$D = 2\mu U \cdot 2\sqrt{\frac{UL}{\nu}} \cdot .332 = 1.328\rho U^2/\sqrt{R}. \tag{8.25}$$

Thus friction drag on a plate is $O(R^{-1/2})$ at large R, at least in a laminar flow.

8.2.2 The Falkner-Skan family of boundary layers

An immediate generalization of the Blasius solution is to boundary layers whose pressure gradient is some power of x. From the Bernoulli equation for steady flow, a gradient $p_x = -mA^2x^{2m-1}$ results from an external stream with velocity $U(x) = Ax^m$. We remark that such a velocity variation occurs on the surface of an infinite wedge aligned with a constant free stream, provided that the half-angle of the wedge is $\frac{m}{m+1}\pi$. Then x is measured along the surface of the wedge, and y is measured perpendicular to the surface. So again there is no length in the problem. The equations to be solved are then, in dimensional form,

$$uu_x + vu_y - mA^2x^{2m-1} - \nu u_{yy} = 0, \quad u_x + v_y = 0. \tag{8.26}$$

Since there is no length, we are led to look for a similarity solution . If we try $\psi = x^\alpha F(y/x^\beta)$, then the factors of x coming from insertion into (8.26) will cancel, leaving an ordinary differential equation, provided that

$$\alpha == \frac{1+m}{2}, \quad \beta = \frac{1-m}{2}. \tag{8.27}$$

Setting

$$\psi = AKx^{\frac{1+m}{2}}F(\eta), \eta = \frac{y}{Kx^{\frac{1-m}{2}}}, K = \sqrt{\frac{\nu}{(m+1)A}}, \tag{8.28}$$

the equation which results (see problem 8.1) is

$$F''' + \frac{1}{2}FF'' + \frac{m}{1+m}(1 - F'^2) = 0. \tag{8.29}$$

The boundary conditions are again $F(0) = F'(0) = 0, F'(\infty) = 1$.

We show in figure 8.4 several profiles for various m. For m positive existence and uniqueness of the solution has be established, and the profiles become somewhat steeper. The cases $m > 0$ are said to correspond to a *favorable pressure gradient*, $U'(x) > 0$ and $p'(x) < 0$. The boundary layer can be said to respond favorably to a pressure which decreases in the streamwise direction. When m becomes negative, the story is significantly different. Uniqueness of the profile can be lost, although profiles such that $u \geq 0$ for all η can be shown to be unique. In figure 8.4 we show the limiting case of such non-negative profiles, occurring when $m = -.0904$. Note that $F''(0) = 0$ for this profile. This implies du/dy vanishes at the wall, and so the viscous friction force is zero there. Positive pressure gradients are said to be *unfavorable*, and can lead to the phenomenon of *boundary layer separation*. We will return to the separation problem below. Here the suggestion is that $m < .0904$ would lead to a boundary layer which has a *negative* value of u_y at the wall, and so would involve a region of reversed flow; the streamline $\psi = 0$ must actually bifurcate from the wall, so the term "separation" is appropriate.

We may summarize the general picture of high Reynolds number flow, as provided by the boundary-layer concept, as follows. For a general finite body in a flow, there should be a portion of the surface of the body, upstream of any

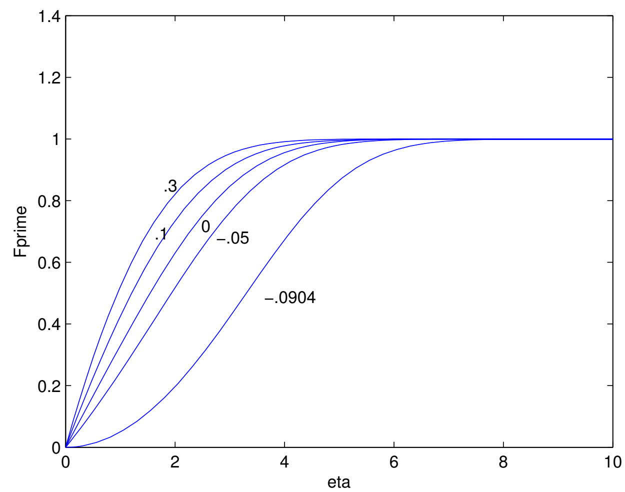

Figure 8.4: The Falkner-Skan profiles for various m.

point or point of separation of the boundary layer, where the flow is that of an inviscid fluid except within a small layer adjacent to the body, called the boundary layer. Within the boundary layer, the pressure gradient is imposed by the inviscid exterior flow. At the same time the boundary layer modifies the inviscid flow slightly due to its displacement thickness. The picture is clouded by separation, and the tendency of high Reynolds number flows to be unstable and hence time dependent.

Finally, with the example which follows we indicate how boundary-layer techniques can arise in a somewhat different context.

Example 8.1

We give here as example of the application of boundary-layer ideas to a

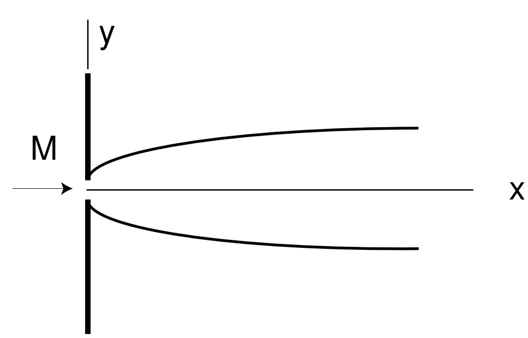

Figure 8.5: A two-dimensional laminar jet emerges from a slit in a wall.

different physical problem. The idea is to model a laminar two-dimensional steady jet issuing from a small slit in a wall, see figure 8.5. We are going to treat the jet as "thin" when $Re \gg 1$, and so apply Prandtl's reasoning to obtain again his boundary layer equations. Since $p_{\bar{y}} = 0$ the is invariant through the jet, and assuming that at $\bar{y} = \infty$ we have uniform conditions, we may assert that p is independent of x, as in Blasius' semi-infinite plate problem. There is no length in the problem (ignoring the small width of the slit), so again we are led to try a solution of the form $\psi = x^\alpha F(y/x^\beta)$. The condition that $\mathbf{u} \cdot \nabla u$ and u_{yy} have common factors of x requires that $\alpha + \beta = 1$. We do not have a nonzero value assumed by F' at infinity, as in the Blasius problem. However there is a new physical constraint. Since the pressure is constant throughout, there are no forces available to cause the net flux of x-momentum to vary as a function of x. Consequently the integral (omitting a constant factor of ρ)

$$M = \int_{-\infty}^{+\infty} u^2 dy \tag{8.30}$$

must be independent of x. This requires that $\beta = 2\alpha$, so that $\alpha = 1/3, \beta = 2/$.[1]

Substituting $\psi = x^{1/3} F(y/x^{2/3})$ into the dimensional equation for ψ,

$$\psi_y \psi_{xy} - \psi_x \psi yy - \nu \psi_{yyy} = 0, \tag{8.31}$$

we get the equation

$$\nu F''' + \frac{1}{3}(FF')' = 0. \tag{8.32}$$

We require that $F'', F' \to 0$ as $\eta \to \infty$ and

$$\int_{-\infty}^{+\infty} F'^2 d\eta = M. \tag{8.33}$$

Integrating twice,

$$\frac{1}{6}F^2 + \nu F' = \frac{1}{6}F_\infty^2, \quad F_\infty = F(\infty). \tag{8.34}$$

The integral yields

$$F = F_\infty \tanh\Big(\frac{F_\infty \eta}{6\nu}\Big). \tag{8.35}$$

Applying the condition (8.33) we obtain

$$\frac{\nu}{4}F_\infty^3 = M, \tag{8.36}$$

which determine F_∞ in terms of M. The velocity component u, which dominates in the jet, is given by

$$u = \frac{F_\infty^2}{6\nu x^{1/3}} \frac{1}{\cosh^2\left(\frac{F_\infty \eta}{6\nu}\right)}. \tag{8.37}$$

[1] To get this by a stretching group, $x \to Ax, y \to By, \psi \to C\psi$, the momentum equation requires $A = BC$ as in the Blasius solution, but the momentum flux constraint is invariant when $C^2 = B$, so then $C^3 = A$. Thus $y/x^{2/3}$ is invariant, and ψ must be proportional to $x^{1/3}$.

Note that the jet spreads as $x^{2/3}$ and decays as $x^{-1/3}$. In practice it is difficult to establish a laminar jet because of instabilities, and the jets obtained in the laboratory are usually turbulent.

8.3 Boundary-layer analysis as a matching problem

We now digress somewhat to indicate some of the mathematical ideas that have grown our of Prandtl's approach to high Reynolds number flow. We have suggested that there is a kind of interaction at work between an "outer", inviscid flow, and an "inner" boundary-layer flow. That is, the pressure gradient is fundamentally an outer condition imposed on the boundary layer. On the other hand the boundary-layer modifies somewhat the strreamlines well away from the body, in the inviscid flow. We now explore a model problem in one space dimension, involving a *singular perturbation* of an ordinary differential equation. The small parameter ϵ will replace $1/Re$, and the problem is not one of fluid dynamics; nevertheless there will be an inner solution and an outer solution that will be analogous to our viscous boundary layer and out outer inviscid flow. We suggest that the model indicates how a more formal approach to boundary layer theory might proceed, although we shall not pursue this further here.

The model problem is the following: let $f(x) = f(x, \epsilon)$ satisfy

$$\epsilon f'' + f' = a, \quad 0 < a < 1, \; 0 \le x \le 1, \quad y(0) = 0, y(1) = 1. \tag{8.38}$$

The "singular" adjective is usually applied to problems where the limiting operation, in this case $\epsilon \to 0$ reduces the order of the differential equation, in our case from order two to order one.

We first define our "outer problem", analogous to the inviscid Euler flow. We bound x away from zero, $0 < A \le x \le 1$, and apply the limit $\epsilon \to 0$ to the model equation. This gives the reduced system

$$f' = a. \tag{8.39}$$

We apply the condition at $x-1$ to the solution of this reduced equation, yielding

$$f_{\text{outer}} = ax + 1 - a. \tag{8.40}$$

We see that f_{outer} does *not* satisfy the condition on f at $x = 0$. This adjustment will happen in a boundary layer near $x = 0$. So we consider with Prandtl how to deal with the combination ϵf_{xx}. If derivatives become large this combination need not be small. On the other hand f_x can also be large, so that it is tempting to suppose that at least minimally ϵf_{xx} and f_x must be the same size. This suggests function of x/ϵ, so we define the stretched variable $\bar{x} = x/\epsilon$. [2]

[2]The fact that we do not have a square root in defining a stretched variable, as we did for the Reynolds number in the Prandtl boundary layer, reflects the vast difference in the fluid equations and the model equation. This however is a relatively unimportant difference.

Using the stretched variable our equation takes the form

$$f_{\bar{x}\bar{x}} + f_{\bar{x}} = \epsilon a. \tag{8.41}$$

We no consider the limit $\epsilon \to 0$ with $0 \le \bar{x} < B < \infty$, obtaining the limiting equation

$$f_{\bar{x}\bar{x}} + f_{\bar{x}} = 0. \tag{8.42}$$

This is our model of the Prandtl boundary layer. We require that its solution vanish at $\bar{x} = 0$, so that

$$f_{\text{inner}} = C(1 - e^{-\bar{x}}). \tag{8.43}$$

Here C is an undetermined constant. Note that $f_{\text{inner}} \to C$ as $\bar{x} \to \infty$, so that we have the model equivalent of obtaining the "velocity at infinity" for the viscous boundary layer. Since f is suppose to be represent by f_{outer} away from $x = 0$, it is natural to identify C with the limit of f_{outer} for small x. This yields

$$C = 1 - a. \tag{8.44}$$

This is usually stated as a *matching condition*:

$$\lim_{\bar{x}\to\infty} f_{\text{inner}} = \lim_{x\to 0} f_{\text{outer}}. \tag{8.45}$$

An approximation to $f(x, \epsilon)$ which applies to the entire interval can be obtained by adding with inner and outer solutions, provided we account for any terms that are common to both. The common part in our problem is just $1 - a$. We define the approximate *composite* solution by

$$f_{comp} = f_{\text{inner}} + f_{\text{outer}} - 1 + a = ax + (1 - a)(1 - e^{-\frac{x}{\epsilon}}). \tag{8.46}$$

It is interesting to compare our approximation with the exact solution of the model problem, namely

$$f(x, \epsilon) = ax + \frac{(1 - a)}{1 - e^{-\frac{1}{\epsilon}}}(1 - e^{-\frac{x}{\epsilon}}). \tag{8.47}$$

The difference is of order $e^{-\frac{1}{\epsilon}}$ uniformly over the domain.

Anyone wishing to explore further the use of singular perturbations in fluid dynamics should consult the book *Perturbation Methods in Fluid Mechanics*, by Milton D. Van Dyke. For boundary-layer theory these methods culminated in an analytical attack on the problem of separation, which we explore briefly in the final section of this chapter.

8.4 Separation

One of the great accomplishments of 20th century fluid dynamics was an understanding of the fundamental mechanisms of separation of a boundary layer in the limit of large R. This work, due to Stewartson, Williams, Messiter, Neiland,

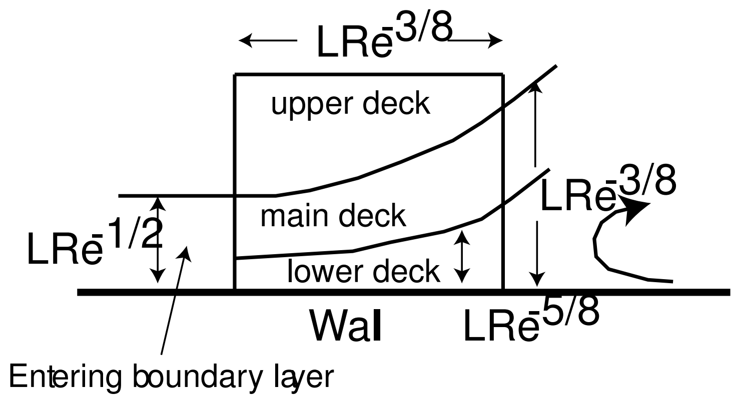

Figure 8.6: The triple deck

Smith, Sychev, Kaplun, and others, led to a full description of the mechanism of separation in a class of problems of wide applicability. A review of much of the work on the separation problem may be found in Stewartwon, K., *D'Alembert's paradox*, SIAM Review vol. 23, No. 3, p.308 (1981).

The main result of this effort has been the so-called *triple deck theory*. The name applies to the layering of domains of different orders of magnitude, in the neighborhood of the point of separation. We show the structure of the triple deck in figure 8.6. The main point to be made in discussing triple deck is that the layered structure results from a *nonlinear interaction between the boundary layer and the pressure gradient*. In other words, separation represents a breaking of the inner-outer separation of the pressure gradient from the boundary layer responding to the pressure gradient. Within the triple-deck region the boundary layer is modifying the pressure gradient, which in turn is affecting the boundary layer. Entering from the left of the main deck is the profile of the boundary layer as it has evolved through a length we call L in the figure. Thus the main deck has thickness $LR^{-1/2}$. The thinner lower deck is a region where the full boundary layer equations apply, with viscous stress important and reversal of the flow occurring following separation. Over a Δx of order $LR^{-3/8}$ the boundary layer is essentially raised by the same order, forming the upper layer. During this lifting the main deck profile is unchanged by viscosity, since it is traversing such a small domain. This lifting of the boundary layer modifies the pressure gradient locally, and this penetrates down to the lower layer, providing the feeedback that completes the cycle.

Unfortunately this brief description of separation does not do justice to the analysis involved, nor to the insight that was needed to determine the construction of the triple deck, nor to the many related questions that have been tackled with this machinery.

Problem set 8

2. Verify (8.27) and (8.29).

2. *Oseen's equations* are sometimes also proposed as a *model* of the Navier-Stokes, equations, in the study of steady viscous flow past a body. Oseen's equations, for a flow with velocity $(U, 0, 0)$ at infinity, are

$$U\frac{\partial \mathbf{u}}{\partial x} + \frac{1}{\rho}\nabla p - \nu\nabla^2\mathbf{u} = 0, \nabla \cdot \mathbf{u} = 0.$$

(a) Show that in this model the vorticity is a function of y, z alone.

(b) For the Oseen model, and for a flat plate aligned with the flow, carry out Prandtl's simplifications for deriving the boundary-layer equations in two dimensions, given that the boundary condition of no slip is retained at the body. That is, find the form of the boundary layer on a flat plate of length L aligned with the flow at infinity, according to Oseen's model, and show that in the boundary layer the the x-component of velocity, u, satisfies

$$U\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial y^2}.$$

What are the boundary conditions on u for the flat-plate problem? Find the solution, by assuming that u is a function of $y\sqrt{\frac{U}{\nu x}}$, for $0 < x < L$.

(c) Compute the drag coefficient of the plate (drag divided by $\rho U^2 L$, and remember there are two sides), in the Oseen model.

3. What are the boundary-layer equations for the boundary-layer on the front portion of a circular cylinder of radius a, when the free stream velocity is $(-U, 0, 0)$? (Use cylindrical polar coordinates). What is the role of the pressure in the problem? Be sure to include the effect of the pressure as an explicit function in your momentum equation, the latter being determined by the potential flow past a circular cylinder studied previously. Show that, by defining $x = a\theta, \bar{y} = (r - a)\sqrt{R}$ in the derivation of the boundary-layer equations, the equations are equivalent to a boundary layer on a flat plate aligned with the free stream, in rectangular coordinates, but with pressure a given function of x.

4. For a *cylindrical* jet emerging from a hole in a plane wall, we have a problem analogous to the 2D jet considered in class. Consider only the boundary-layer limit. (a) Show that

$$\frac{\partial}{\partial z}(u_z^2) + \frac{1}{r}\frac{\partial}{\partial r}(ru_r u_z) - \frac{\nu}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}) = 0,$$

and hence that the momentum M is a constant, where

$$M = 2\pi\rho \int_0^\infty r u_z^2 dr.$$

(b) Letting $(u_z, u_r) = (1/r)(\psi_r, -\psi_z)$ where $\psi(0, z) = 0$ show that we must have $\psi = zf(\eta),\ \eta = r^2/z^2$. Determine the equation for f and thus show that the boundary-layer limit has the form

$$f = 4\nu \frac{\eta}{\eta + \eta_0},$$

where η_0 is a constant. Express η_0 in terms of M, the momentum flux of the jet defined above.

5.consider the Prandtl boundary-layer equations with $U(x) = 1/x$, so $p(x)/\rho = p_\infty - 1/(2x^2)$. Verify that the similarity solution has the form $\psi = f(\eta), \eta = y/x$. Find the equation for f. Show that there is no continuously differentiable solution of the equation which satisfies $f(0) = f'(0) = 0$ and $f' \to 1, f'' \to 0$ as $\eta \to \infty$. (Hint: Obtain an equation for $g = f'$.)