An Introduction to Theoretical Fluid Dynamics

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Notes on the course

The course meets on Wednesday, 9:20-11:30 am in Room 813 WWH.
Office Hours: TBA

BOOKS: The text is Landau and Lifshitz, Fluid Mechanics, available at the Bookstore. Acheson's Elementary Fluid Dynamics is a good elementary book. The reserve books given below will be available in the Courant library.

- Batchelor, G.K. Introduction to Fluid Dynamics, Cambridge Univsrsity Press 1967
- Landau and Lifshitz, Fluid Mechanis (2nd Ed.), Pergamon Press 1987.
- Milne-Thomson, L.M. Theoretical Hydrodynamics, McMillan (5th Ed.)
- Lighthill, M.J. An Informal Introduction to Theoretical Fluid Mechanics, Clarendon Press 1986.
- Prandtl, L. Essentials of Fluid Dynamics, Hafner 1952.
- Lamb, Hydrodynamics (6th Ed.), Cambridge University Press 1932
- Courant and Freidrichs, Supersonic Flow and Shock Waves, Interscience 1948.
- Meyer, An Introduction to Mathematical Fluids Dynamics, Dover 1971.
- D. J. Acheson, Elementary Fluid Dynamics, Clarendon 1990.

Chapter 1

The fluid continuum

This course will deal with a mathematical idealization of common fluids such as air or water. The main idealization is embodied in the notion of a *continuum* and our "fluids" will generally be identified with a certain connected set of points in R^N, where we will consider dimension N to be 1,2, or 3. Of course the fluids will move, so basically our subject is that of a moving continuum.

This description is an idealization which neglects the molecular structure of real fluids. *Liquids* are fluids characterized by random motions of molecules on the scale of $10^{-7} - 10^{-8}$ cm, and by a substantial resistance to compression. *Gases* consist of molecules moving over much larger distances, with mean free paths of the order of 10^{-3} cm, and are readily compressed. Both liquids and gases will fall within the scope of the theory of fluid motion which we will develop below. The theory will deal with observable properties such as velocity, density, and pressure. These properties must be understood as averages over volumes which contains many molecules but are small enough to be "infinitesimal" with respect to the length scale of variation of the property. We shall use the term *fluid parcel* to indicate such a small volume. The notion of a *particle* of fluid will also be used, but should not be confused with a molecule. For example, the time rate of change of position of a fluid particle will be the *fluid velocity*, which is an average velocity taken over a parcel and is distinct from molecular velocities. The continuum theory has wide applicability to the natural world, but there are certain situations where it is not satisfactory. Usually these will involve small domains where the molecular structure becomes important, such as shock waves or fluid interfaces.

1.1 Eulerian and Lagrangian descriptions

Let the independent variables (observables) describing a fluid be a function of position $\mathbf{x} = (x_1, \ldots, x_N)$ in Euclidean space and time t. Suppose that at $t = 0$ the fluid is identified with an open set $\mathcal{S}_0$ of R^N. As the fluid moves, the particles of fluid will take up new positions, occupying the set $\mathcal{S}_t$ at time

t. We can introduce the map $\mathcal{M}_t, \mathcal{S}_0 \to \mathcal{S}_t$ to describe this change, and write $\mathcal{M}_t\mathcal{S}_0 = \mathcal{S}_t$. If $\mathbf{a} = (a_1, \ldots, a_N)$ is a point of $\mathcal{S}_0$, we introduce the function $\mathbf{x} = \mathcal{X}(\mathbf{a}, t)$ as the position of a fluid particle at time t, which was located at $\mathbf{a}$ at time $t = 0$. The function $\mathcal{X}(\mathbf{a}, t)$ is called the *Lagrangian coordinate* of the fluid particle identified by the point $\mathbf{a}$. We remark that the"coordinate" $\mathbf{a}$ need not in fact be the initial position of a particle, although that is the most common choice and will be generally used here. But any unique labeling of the particles is acceptable.[1]

The *Lagrangian description* of a fluid emerges from this focus on the fluid properties associated with individual fluid particles. To "think Lagrangian" about a fluid, one must move with the fluid and sample the fluid properties in each moving parcel. The Lagrangian analysis of a fluid has certain conceptual and mathematical advantages, but it is often difficult to apply to useful examples. Also it is not directly related to experience, since measurements in a fluid tend to be performed at fixed points in space, as the fluid flows past the point.

If we therefore adopt the point of view that we will observe fluid properties at a fixed point $\mathbf{x}$ as a function of time, we must break the association with a given fluid particle and realize that as time flows different fluid particles will occupy the position $\mathbf{x}$. This will make sense as long as $\mathbf{x}$ remains within the set $\mathcal{S}_t$. Once properties are expressed as functions of $\mathbf{x}, t$ we have the *Eulerian description* of a fluid. For example, we might consider the fluid to fill all space and be at rest "at infinity". We then can consider the velocity $\mathbf{u}(\mathbf{x}, t)$ at each point of space, with $\lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x}, t) = 0$. Or, we might have a fixed rigid body with fluid flowing over it such that at infinity we have a fixed velocity $\mathbf{U}$. For points outside the body the fluid velocity will be defined and satisfy $\lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x}, t) = \mathbf{U}$.

It is of interest to compare these two descriptions of a fluid and understand their connections. The most obvious is the meaning of velocity: the definition is

$$\mathbf{x}_t = \frac{\partial \mathcal{X}}{\partial t}\Big|_{\mathbf{a}} = \mathbf{u}(\mathbf{x}(\mathbf{a}, t), t). \tag{1.1}$$

That is to say, following the particle we calculate the rate of change of position with respect to time. Given the Eulerian velocity field, the calculation of Lagrangian coordinates is therefore mathematically equivalent to solving the initial-value problem for the system (1.1) of ordinary differential equations for the function $\mathbf{x}(t)$, with the initial condition $\mathbf{x}(0) = \mathbf{a}$, the order of the system being the dimension of space. The special case of a steady flow leads to a system of *autonomous* ODEs.

Example 1.1: In two dimensions ($N = 2$), with fluid filling the plane, we take $\mathbf{u}(\mathbf{x}, t) = (u(x, y, t), v(x, y, t)) = (x, -y)$. This velocity field is independent of time, hence we call it a *steady flow*. To compute the Lagrangian coordinates of the fluid particle initially at $\mathbf{a} = (a, b)$ we solve:

$$\frac{\partial x}{\partial t} = x, x(0) = a, \quad \frac{\partial y}{\partial t} = -y, y(0) = b, \tag{1.2}$$

[1] We shall often use (x, y, z) in place of (x_1, x_2, x_3), and (a, b, c) in place of (a_1, a_2, a_3).

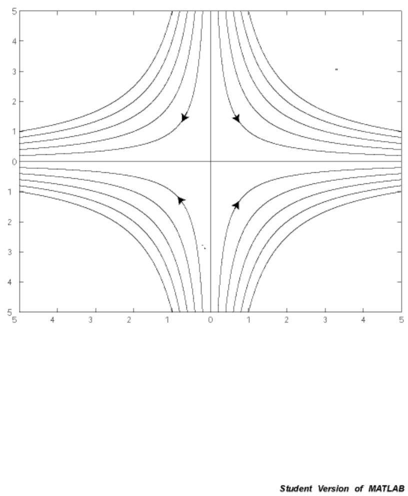

Figure 1.1: Stagnation-point flow

so that $\mathcal{X} = (ae^t, be^{-t})$. Note that, since $xy = ab$, the *particle paths* are hyperbolas; the curves independent of time, see figure 1.1. If we consider the fluid in $y > 0$ only and take $y = 0$ as a rigid wall, we have a flow which is impinging vertically on a wall. The point $x = y = 0$, where the velocity is zero, is called a *stagnation point*. This point is a hyperbolic point relative to particle paths. A flow of this kind occurs at the nose of a smooth body placed in a uniform current. Because this flow is steady, the hyperbolic particle paths are also called *streamlines*.

Example 1.2: Again in two dimensions, consider $(u, v) = (y, -x)$. Then $\frac{\partial x}{\partial t} = y$ and $\frac{\partial y}{\partial t} = -x$. Solving, the Lagrangian coordinates are $x = a\cos t + b\sin t, y = -a\sin t + b\cos t$, and the particle paths (and streamlines) are the circles $x^2 + y^2 = a^2 + b^2$. The motion on the streamlines is clockwise, and fluid particles located at some time on a ray x/y =constant remain on the same ray as it rotates clockwise once for every 2π units of time. This is *sold-body rotation*.

Example 1.3: If instead $(u, v) = (y/r^2, -x/r^2), r^2 = x^2 + y^2$, we again have particle paths which are circles, but the velocity becomes infinite at $r = 0$. This is an example of a flow representing a *point vortex*. We shall take up the study of vortices in chapter 3.

1.1.1 Particle paths, instantaneous streamlines, and streak lines

The present considerations are *kinematic*, meaning that we are assuming knowledge of fluid motion, through an Eulerian velocity field $\mathbf{u}(\mathbf{x}, t)$ or else Lagrangian coordinates $\mathbf{x} = \mathcal{X}(\mathbf{a}, t)$, irrespective of the cause of the motion. One useful kinematic characterization of a fluid flow is the pattern of streamlines, as al-

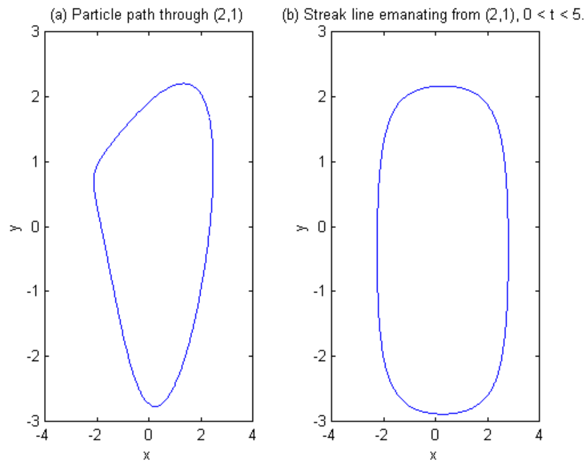

Figure 1.2: Particle path and streak line in example 1.4.

ready mentioned in the above examples. In steady flow the streamlines and particle paths coincide. In an unsteady flow this is not the case and the only useful recourse is to consider *instantaneous streamlines*, at a particular time. In three dimensions the instantaneous streamlines are the orbits of the $\mathbf{u}(\mathbf{x}, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$ at time t. These are the integral curves satisfying

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{1.3}$$

As time flows these streamlines will change in an unsteady flow, and the connection with particle paths is not obvious in flows of any complexity.

Visualization of flows in water is sometimes accomplished by introducing dye at a point in space. The dye can be thought of as labeling by color the fluid particle found at the point at a given time. As each point is labeled it moves along its particle path. The resulting *streak line* thus consists of all particles which at some time in the past were located at the point of injection of the dye. To describe a streak line mathematically we need to generalize the time of initiation of a particle path. Thus we introduce the *generalized Lagrangian coordinate* $\mathbf{x} = \mathcal{X}(\mathbf{a}, t, t_a)$, defined to be the position at time t of a particle that was located at $\mathbf{a}$ at time t_a. A streak line observed at time $t > 0$, which was started at time $t = 0$ say, is given by $\mathbf{x} = \mathcal{X}(\mathbf{a}, t, t_a), 0 < t_a < t$. Particle paths, instantaneous streamlines, and streak lines are all distinct objects in unsteady flows.

Example 1.4: Let $(u, v) = (y, -x + \epsilon \cos \omega t)$. For this flow the instantaneous streamlines satisfy $dx/y = dy/(-x + \epsilon \cos \omega t)$ and so are the circles $(x - \epsilon \cos \omega t)^2 + y^2 =$ constant. The generalized Lagrangian coordinates can be obtained from the general solution of a second-order ODE and takes the form

$$x = -\frac{\epsilon}{\omega^2-1}\cos\omega t + A\cos t + B\sin t,\ y = \frac{\epsilon\omega}{\omega^2-1}\sin\omega t + B\cos t - A\sin t, \quad (1.4)$$

where

$$A = -b\sin t_a + \frac{\epsilon\omega}{\omega^2-1}\sin\omega t_a \sin t_a + a\cos t_a + \frac{\epsilon}{\omega^2-1}\cos\omega t_a\cos t_a, \quad (1.5)$$

$$B = a\sin t_a + b\cos t_a - \frac{\epsilon}{\omega^2-1}\cos\omega t_a\sin t_a + \frac{\epsilon\omega}{\omega^2-1}\sin\omega t_a\cos t_a. \quad (1.6)$$

The particle path with $t_a = 0, \omega = 2, \epsilon = 1$ starting at the point $(2,1)$ is given by

$$x = -\frac{1}{3}\cos 2t + \sin t + \frac{7}{3}\cos t, y = \cos t - \frac{7}{3}\sin t + \frac{2}{3}\sin 2t, \quad (1.7)$$

and is shown in figure 1.2(a). All particle paths are closed curves. The streak line emanating from (2,1) over the time interval $0 < t < 2\pi$ is shown in figure 1.2(b).

This last example is especially simple since the 2D system is linear and integrable explicitly. In general two-dimensional unsteady flows and three-dimensional steady flows can exhibit chaotic particle paths and streak lines.

Example 1.5: A nonlinear system exhibiting this complex behavior is the oscillating point vortex: $(u, v) = (y/r^2, -(x-\epsilon\cos\omega t)/r^2)$. We show an example of particle path and streak line in figure 1.3.

1.1.2 The Jacobian matrix

We will, with a few obvious exceptions, be taking all of our functions as infinitely differentiable wherever they are defined. In particular we assume that Lagrangian coordinates will be continuously differentiable with respect to the particle label $\mathbf{a}$. Accordingly we may define the Jacobian of the Lagrangian map $\mathcal{M}_t$ by matrix

$$J_{ij} = \frac{\partial x_i}{\partial a_j}\Big|_t \quad (1.8)$$

Thus $dl_i = J_{ij}da_j$ is a differential vector which can be visualized as connecting two nearby fluid particles whose labels differ by da_j.[2] If $da_1\cdots da_N$ is the volume of a small fluid parcel, then $\mathrm{Det}(\mathbf{J})da_1\cdots a_N$ is the volume of that parcel under the map $\mathcal{M}_t$. Fluids which are *incompressible* must have the property that all fluid parcels preserve their volume, so that Det($\mathbf{J}$)= constant=1 when $\mathbf{a}$ denotes initial position, independently of $\mathbf{a}, t$. We then say that the Lagrangian map is volume preserving. For general compressible fluids Det($\mathbf{J}$) will vary in space and time.

Another important assumption that we shall make is that the map $\mathcal{M}_t$ is always invertible, Det($\mathbf{J}$)> 0. Thus when needed we can invert to express $\mathbf{a}$ as a function of $\mathbf{x}, t$.

[2]Here and elsewhere the summation convention is understood: unless otherwise started repeated indices are to be summed from 1 to N.

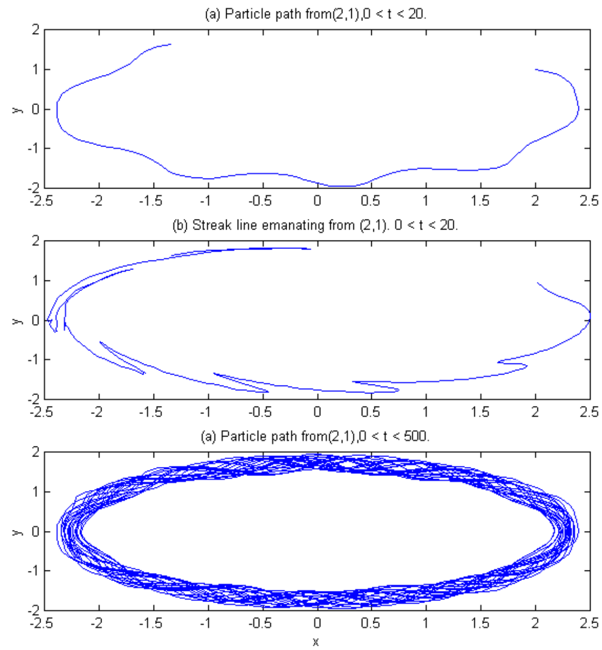

Figure 1.3: The oscillating vortex, $\epsilon = 1.5, \omega = 2$.

1.2 The material derivative

Suppose we have some scalar property $\mathcal{P}$ of the fluid that can be attached to a certain fluid parcel, e.g. temperature or density. Further, suppose that, as the parcel moves, this property is invariant in time. We can express this fact by the equation

$$\frac{\partial \mathcal{P}}{\partial t}\Big|_{\mathbf{a}} = 0, \tag{1.9}$$

since this means that the time derivative is taken with particle label fixed, i.e. taken as we move with the fluid particle in question. We will say that such an invariant scalar is *material*. A material invariant is one attached to a fluid particle. We now asked how this property should be expressed in Eulerian variables. That is, we select a point $\mathbf{x}$ in space and seek to express material invariance in terms of properties of the fluid *at this point*. Since the fluid is generally moving at the point, we need to bring in the velocity. The way to do this is to differentiate $\mathcal{P}(\mathbf{x}(\mathbf{a}, t), t)$, expressing the property as an Eulerian variable, using the chain rule:

$$\frac{\partial \mathcal{P}(\mathbf{x}(\mathbf{a}, t), t)}{\partial t}\Big|_{\mathbf{a}} = 0 = \frac{\partial \mathcal{P}}{\partial t}\Big|_{\mathbf{x}} + \frac{\partial x_i}{\partial t}\Big|_{\mathbf{a}} \frac{\partial \mathcal{P}}{\partial x_i}\Big|_{t} = \mathcal{P}_t + \mathbf{u} \cdot \nabla \mathcal{P}. \tag{1.10}$$

In fluid dynamics the Eulerian operator $\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla$ is called the *material derivative* or *substantive* derivative or *convective* derivative. Clearly it is a time derivative "following the fluid", and translates the Lagrangian time derivative in terms of Eulerian properties of the fluid.

Example 1.6: The *acceleration* of a fluid parcel is defined as the material derivative of the velocity $\mathbf{u}$. In Lagrangian variables the acceleration is $\frac{\partial^2\mathbf{x}}{\partial t^2}\Big|_{\mathbf{a}}$, and in Eulerian variables the acceleration is $\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}$.

Following a common convention we shall often write

$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla, \tag{1.11}$$

so the acceleration becomes $D\mathbf{u}/Dt$.

Example 1.7: We consider the material derivative of the determinant of the Jacobian $\mathbf{J}$. We may divide up the derivative of he determinant into a sum of N determinants, the first having the first row differentiated, the second having the next row differentiated, and so on. The first term is thus the determinant of the matrix

$$\begin{pmatrix} \frac{\partial u_1}{\partial a_1} & \frac{\partial u_1}{\partial a_2} & \cdots & \frac{\partial u_1}{\partial a_N} \\ \frac{\partial x_2}{\partial a_1} & \frac{\partial x_2}{\partial a_2} & \cdots & \frac{\partial x_2}{\partial a_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_N}{\partial a_1} & \frac{\partial x_N}{\partial a_2} & \cdots & \frac{\partial x_N}{\partial a_N} \end{pmatrix}. \tag{1.12}$$

If we expand the terms of the first row using he chain rule, e.g.

$$\frac{\partial u_1}{\partial a_1}=\frac{\partial u_1}{\partial x_1}\frac{\partial x_1}{\partial a_1}+\frac{\partial u_1}{\partial x_2}\frac{\partial x_2}{\partial a_1}+\cdots+\frac{\partial u_1}{\partial x_N}\frac{\partial x_N}{\partial a_1}, \tag{1.13}$$

we see that we will get a contribution only from the terms involving $\frac{\partial u_1}{\partial x_1}$, since all other terms involve the determinant of a matrix with two identical rows. Thus the term involving the derivative of the top row gives the contribution $\frac{\partial u_1}{\partial x_1}\mathrm{Det}(\mathbf{J})$. Similarly, the derivatives of the second row gives the additive contribution $\frac{\partial u_2}{\partial x_2}\mathrm{Det}(\mathbf{J})$. Continuing, we obtain

$$\frac{D}{Dt}\mathrm{Det}\mathbf{J}=\mathrm{div}(\mathbf{u})\ \mathrm{Det}(\mathbf{J}). \tag{1.14}$$

Note that, since an incompressible fluid has $\mathrm{Det}(\mathbf{J})=1$, such a fluid must satisfy, by (1.14), $\mathrm{div}(\mathbf{u})=0$, which is the way an incompressible fluid is defined in Eulerian variables.

1.2.1 Solenoidal velocity fields

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The adjective *solenoidal* applied to a vector field is equivalent to "divergence-free". We will used either $\mathrm{div}(\mathbf{u})$ or $\nabla\cdot\mathbf{u}$ to denote divergence. The incompressibility of a material with a solenoidal vector field means that the Lagrangian

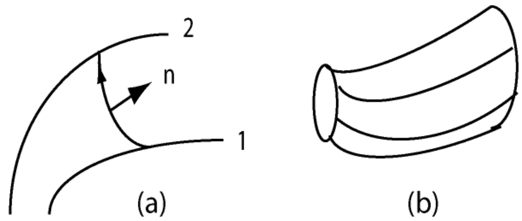

Figure 1.4: Solenoidal velocity fields. (a) Two streamlines in two dimensions. (b) A stream tube in three dimensions.

map $\mathcal{M}_t$ preserves volume and so whatever fluid moves into a region of space is matched by an equal amount of fluid moving out. In two dimensions the equation expressing the solenoidal condition is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.15}$$

If $\psi(x, y)$ posses continuous second derivatives we may satisfy (1.15) by setting

$$u = \frac{\partial \psi}{\partial y}. \quad v = -\frac{\partial \psi}{\partial x}. \tag{1.16}$$

The function ψ is called the *stream function* of the velocity field. The reason for the term is immediate: The instantaneous streamline passing through x, y has direction $(u(x, y), v(x, y))$ at this point. The normal to the streamline at this point is $\nabla\psi(x, y)$. But we see from (1.16) that $(u, v) \cdot \nabla\psi = 0$ there, so the lines of constant ψ are the instantaneous streamlines of (u, v).

Consider two streamlines $\psi = \psi_i, i = 1, 2$ and any oriented simple contour (no self-crossings) connecting one streamline to the other. The claim is then that the flux of fluid across this contour, from left to right seen by an observed facing in the direction of orientation of the contour is given by the difference of the values of the stream function, $\psi_2 - \psi_1$ if the contour is oriented to go from streamline 1 to streamline 2, see figure 1.4(a). Indeed, oriented as shown the line integral of flux is just $\int (u, v) \cdot (dy, -dx) = \int d\psi = \psi_2 - \psi_1$. In three dimensions, we similarly introduce a *stream tube*, consisting of a collection of streamlines, see figure (1.4)(b). The flux of fluid across any "face" cutting through the tube must be the same. This follows immediately by applying the divergence theorem to the integral of div $\mathbf{u}$ over the stream tube. Note that we are referring here to the flux of volume of fluid, not flux of mass.

In three dimensions there are various "stream functions" used when special symmetry allow them. An example of a class of solenoidal flows generated by two scalar functions is $\mathbf{u} = \nabla\alpha \times \nabla\beta$ where the intersections of the surfaces of constant $\alpha(x, y, z)$ and $\beta(x, y, z)$ are the streamlines. Since $\nabla\alpha \times \nabla\beta = \nabla \times (\alpha\nabla\beta)$ we see that these flows are indeed solenoidal.

1.2.2 The convection theorem

Suppose that $\mathcal{S}_t$ is a region of fluid particles and let $f(\mathbf{x}, t)$ be a scalar function. Forming the volume integral over $\mathcal{S}_t$, $F = \int_{\mathcal{S}_t} f dV_{\mathbf{x}}$, we seek to compute $\frac{dF}{dt}$. Now $dV_{\mathbf{x}} = dx_1 \cdots dx_N = \mathrm{Det}(\mathbf{J}) da_1 \cdots a_N = \mathrm{Det}(\mathbf{J}) dV_{\mathbf{a}}$. Thus

$$\frac{dF}{dt} = \frac{d}{dt}\int_{\mathcal{S}_0} f(\mathbf{x}(\mathbf{a},t),t)\mathrm{Det}(\mathbf{J})dV_{\mathbf{a}} = \int_{\mathcal{S}_0} \mathrm{Det}(\mathbf{J})\frac{d}{dt} f(\mathbf{x}(\mathbf{a},t),t)dV_{\mathbf{a}}$$

$$+\int_{\mathcal{S}_0} f(\mathbf{x}(\mathbf{a},t),t)\frac{d}{dt}\mathrm{Det}(\mathbf{J})dV_{\mathbf{a}} = \int_{\mathcal{S}_0}\Big[\frac{Df}{Dt} + f\mathrm{div}(\mathbf{u})\Big]\mathrm{Det}(\mathbf{J})dV_{\mathbf{a}},$$

and so

$$\frac{dF}{dt} = \int_{\mathcal{S}_t}\Big[\frac{Df}{Dt} + f\mathrm{div}(\mathbf{u})\Big]dV_{\mathbf{x}}. \tag{1.17}$$

The result (1.17) is called the *convection theorem*. We can contrast this calculation with one over a fixed finite region $\mathcal{R}$ of space with boundary $\partial\mathcal{R}$. In that case the rate of change of f contained in $\mathcal{R}$ is just

$$\frac{d}{dt}\int_{\mathcal{R}} f dV_{\mathbf{x}} = \int_{\mathcal{R}} \frac{\partial f}{\partial t} dV_{\mathbf{x}}. \tag{1.18}$$

The difference between the two calculations involves the *flux* of f through the boundary of the domain. Indeed we can write the convection theorem in the form

$$\frac{dF}{dt} = \int_{\mathcal{S}_t}\Big[\frac{\partial f}{\partial t} + \mathrm{div}(f\mathbf{u})\Big]dV_{\mathbf{x}}. \tag{1.19}$$

Using the divergence (or Gauss') theorem, and considering the instant when $\mathcal{S}_t = \mathcal{R}$, we have

$$\frac{dF}{dt} = \int_{\mathcal{R}} \frac{\partial f}{\partial t} dV_{\mathbf{x}} + \int_{\partial\mathcal{R}} f\mathbf{u}\cdot\mathbf{n} dS_{\mathbf{x}}, \tag{1.20}$$

where $\mathbf{n}$ is the outer normal to the region and $dS_{\mathbf{x}}$ is the area element of $\partial\mathcal{R}$. The second term on the right is flux of f out of the region $\mathcal{R}$. Thus the convection theorem incorporates into the change in f within a region, the flux of f into or out of the region, due to the motion of the boundary of the region. Once we identify f with a useful physical property of the fluid, the convection theorem will be useful for expressing the *conservation* of this property, see chapter 2.

1.2.3 Material vector fields: The Lie derivative

Certain vector fields in fluid mechanics, and notably the *vorticity field*, $\boldsymbol{\omega}(\mathbf{x}, t) = \nabla \times \mathbf{u}$, see chapter 3, can in certain cases behave as a *material vector field*. To understand the concept of a material vector one must imagine the direction of the vector to be determined by nearby material points. It is wrong to think of a material vector as attached to a fluid particle and constant there. This would amount to a simple translation of the vector along the particle path.

Instead, we want the direction of the vector to be that of a differential segment connecting two nearby fluid particles, $dl_i = J_{ij}da_j$. Furthermore, the length of the material vector is to be proportional to this differential length as time evolves and the particles move. Consequently, once the particles are selected, the future orientation and length of a material vector will be completely determined by the Jacobian matrix of the flow.

Thus we define a material vector field as one of the form (in Lagrangian variables)

$$v_i(\mathbf{a}, t) = J_{ij}(\mathbf{a}, t)V_j(\mathbf{a}) \tag{1.21}$$

Of course, given the inverse $\mathbf{a}(\mathbf{x}, t)$ we can express v as a function of $\mathbf{x}, t$ to obtain its Eulerian structure.

We now determine the time rate of change of a material vector field following the fluid parcel. To obtain this we differentiate $v(\mathbf{a}, t)$ with respect to time for fixed $\mathbf{a}$, and develop the result using the chain rule:

$$\begin{aligned} \frac{\partial v_i}{\partial t}\bigg|_{\mathbf{a}} &= \frac{\partial J_{ij}}{\partial t}\bigg|_{\mathbf{a}} V_j(\mathbf{a}) = \frac{\partial u_i}{\partial a_j}V_j \\ &= \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial a_j}V_j = v_k\frac{\partial u_i}{\partial x_k}. \end{aligned} \tag{1.22}$$

Introducing the material derivative, we see that a material vector field satisfies the following equation in Eulerian variables:

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t}\bigg|_{\mathbf{x}} + \mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{u} \equiv v_t + \mathcal{L}_{\mathbf{u}}\mathbf{v} = 0 \tag{1.23}$$

In differential geometry $\mathcal{L}_{\mathbf{u}}$ is called the Lie derivative of the vector field $\mathbf{v}$ with respect to the vector field $\mathbf{u}$.

The way this works can be understood by moving neighboring point along particle paths.

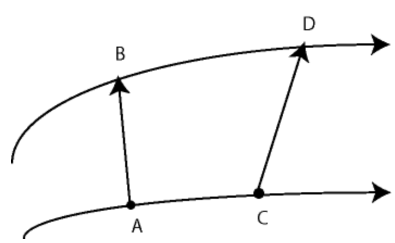

Figure 1.5: Computing the time derivative of a material vector.

Let $\mathbf{v} = \overline{AB} = \Delta\mathbf{x}$ be a small material vector at time t, see figure 1.5. At time Δtlater, the vector has become $\overline{CD}$. The curved lines are the particle paths through A, B of the vector field $\mathbf{u}(\mathbf{x}, t)$. Selecting A as $\mathbf{x}$, we see that after a small time interval Δt the point C is $A + \mathbf{u}(\mathbf{x}, t)\Delta t$ and D is the point $B + \mathbf{u}(\mathbf{x} + \Delta\mathbf{x}, t)\Delta t$. Consequently

$$\frac{\overline{CD} - \overline{AB}}{\Delta t} = \mathbf{u}(\mathbf{x} + \Delta\mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t). \tag{1.24}$$

The left-hand side of (1.24) is approximately $D\mathbf{v}/Dt$, and right-hand side is approximately $\mathbf{v} \cdot \nabla \mathbf{u}$, so in the line $\Delta \mathbf{x}, \Delta t \to 0$ we get (1.23). A material vector field has the property that its magnitude can change by the stretching properties of the underlying flow, and its direction can change by the rotation of the fluid parcel.

Problem Set 1

1. Consider the flow in the (x, y) plane given by $u = -y, v = x + t$. (a) What is the instantaneous streamline through the origin at $t = 1$?(b) what is the path of the fluid particle initially at the origin, $0 < t < 6\pi$? (c) What is the streak line emanating form the origin, $0 < t < 6\pi$?

2. Consider the "point vortex " flow in two dimensions,

$$(u, v) = UL(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}),\ x^2 + y^2 \neq 0,$$

where U, L are reference values of speed and length. (a) Show that the Lagrangian coordinates for this flow may be written

$$x(a, b, t) = R_0 \cos(\omega t + \theta_0),\ y(a, b, t) = R_0 \sin(\omega t + \theta_0)$$

where $R_0^2 = a^2 + b^2, \theta_0 = \arctan(b/a)$, and $\omega = UL/R_0^2$. (b) Consider, at $t = 0$ a small rectangle of marked fluid particles determined by the points $A(L, 0), B(L + \Delta x, 0), C(L + \Delta x, \Delta y), D(L, \Delta y)$. If the points move with the fluid, once point A returns to its initial position what is the shape of the marked region? Since $(\Delta x, \Delta y)$ are small, you may assume the region remains a parallelogram. Do this, first, by computing the entry $\partial y/\partial a$ in the Jacobian, evaluated at $A(L, 0)$. Then verify your result by considering the "lag" of particle B as it moves on a slightly larger circle at a slightly slower speed, relative to particle A, for a time taken by A to complete one revolution.

3. As was noted in class, Lagrangian coordinates can use any unique labeling of fluid particles. To illustrate this, consider the Lagrangian coordinates in two dimensions

$$x(a, b, t) = a + \frac{1}{k} e^{kb} \sin k(a + ct),\ y = b - \frac{1}{k} e^{kb} \cos k(a + ct),$$

where k, c are constants. Note here a, b are *not* equal to (x, y) for any t_0 By examining the determinant of the Jacobian, verify that this gives a unique labeling of fluid particles provided that $b \neq 0$. What is the situation if $b = 0$?(These waves, which were discovered by Gerstner in 1802, represent gravity waves if $c^2 = g/k$ where g is the acceleration of gravity. They do not have any simple Eulerian representation. These waves are discussed in Lamb's book.)

4. In one dimension, the Eulerian velocity is given to be $u(x, t) = 2x/(1 + t)$.

(a) Find the Lagrangian coordinate $x(a,t)$. (b) Find the Lagrangian velocity as a function of a,t. (c) Find the Jacobean $\partial x/\partial a = J$ as a function of a,t.

5. For the stagnation-point flow $\mathbf{u} = (u,v) = U/L(x,-y)$, show that a fluid particle in the first quadrant which crosses the line $y = L$ at time $t = 0$, crosses the line $x = L$ at time $t = \frac{L}{U}\log(UL/\psi)$ on the streamline $Uxy/L = \psi$. Do this in two ways. First, consider the line integral of $\mathbf{u}\cdot\vec{ds}/(u^2+v^2)$ along a streamline. Second, use Lagrangian variables.

6. Let S be the surface of a deformable body in three dimension, and let $I = \int_S f\mathbf{n}dS$ for some scalar function f, $\mathbf{n}$ being the outward normal. Show that

$$\frac{d}{dt}\int f\mathbf{n}dS = \int_S \frac{\partial f}{\partial t}\mathbf{n}dS + \int_S (\mathbf{u}_b\cdot\mathbf{n})\nabla f dS. \tag{1.25}$$

(Hint: First convert to a volume integral between S and an outer surface S' which is *fixed*. Then differentiate and apply the convection theorem. Finally convert back to a surface integral.)

Chapter 2

Conservation of mass and momentum

2.1 Conservation of mass

Every fluid we consider is endowed with a non-negative *density*, usually denoted by ρ, which is in the Eulerian setting is a scalar function of $\mathbf{x}, t$. Its unit are mass per unit volume. Water has a density of about 1 gram per cubic centimeter. For air the density is about 10^{-3} grams per cubic centimeter, but of course pressure and temperature affect air density significantly. The air in a room of a thousand cubic meters= 10^9 cubic centimeters weighs about a thousand kilograms, or more than a ton!

2.1.1 Eulerian form

Let us suppose that mass is being added or subtracted from space as a function $q(\mathbf{x}, t)$, of dimensions mass per unit volume per unit time. The conservation of mass in a fixed region $\mathcal{R}$ can be expressed using (1.20) with $f = \rho$:

$$\frac{d}{dt}\int_{\mathcal{R}} \rho dV_{\mathbf{x}} = \int_{\mathcal{R}} \frac{\partial \rho}{\partial t} dV_{\mathbf{x}} + \int_{\partial\mathcal{R}} \rho \mathbf{u} \cdot \mathbf{n} dS_{\mathbf{x}}. \tag{2.1}$$

Now

$$\frac{d}{dt}\int_{\mathcal{R}} \rho dV_{\mathbf{x}} = \int_{\mathcal{R}} q dV_{\mathbf{x}} \tag{2.2}$$

and if we bring the surface integral in (2.1) back into the volume integral using the divergence theorem we arrive at

$$\int_{\mathcal{R}} \Big[\frac{\partial \rho}{\partial t} + \mathrm{div}(\mathbf{u}\rho) - q\Big] dV_{\mathbf{x}} = 0. \tag{2.3}$$

Since our functions are continuous and $\mathcal{R}$ is an arbitrary open set in R_N, the integrand in (2.3) must vanish, yielding the conservation of mass equation in the Eulerian form:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\mathbf{u}\rho) = q. \tag{2.4}$$

Note that this last equation can also be written

$$\frac{D\rho}{Dt} + \rho \mathrm{div}\, \mathbf{u} = q. \tag{2.5}$$

The conservation of mass equation in either of these forms is sometimes called (for obscure reasons) the *equation of continuity*.

The form (2.5) shows that the material derivative of the density changes in two ways, either by sources and sinks of mass $q > 0$ or $q < 0$ respectively, or else by the non-vanishing of the divergence of the velocity field. A positive value of the divergence, as for $\mathbf{u} = (x, y, z)$, is associated with an expansive flow, thereby reducing local density. This can be examined more closely as follows. Let V be a small volume of fluid where the density is essentially constant. Then ρV is the mass within this fluid parcel, which is a material invariant $D(\rho V)/Dt = 0$. Thus $D\rho/Dt + \rho V^{-1} DV/Dt = 0$. Comparing this with (2.5) we have

$$\mathrm{div}\, \mathbf{u} = \frac{1}{V}\frac{DV}{Dt}. \tag{2.6}$$

Example 2.1: As we have seen in Chapter 1, an incompressible fluid satisfies div $\mathbf{u} = 0$. For such a fluid, free of sources or sinks of mass, we have

$$\frac{D\rho}{Dt} = 0, \tag{2.7}$$

that is, now density becomes a material property. This does not say that the density is constant everywhere in space, only that is constant at a given fluid parcel, as it moves about . (Note that we use parcel here to suggest that we have to average over a small volume to compute the density.) However a fluid of constant density without mass addition *must* be incompressible. This difference is important. Sea water is essentially incompressible but density changes due to salinity are an important part of the dynamics of the oceans.

2.1.2 Lagrangian form

If $q = 0$ the Lagrangian form of the conservation of mass is very simple because if we move with the fluid the density changes that we see are due to expansion and dilation of the fluid parcel, which is controlled by Det($\mathbf{J}$). Let a parcel have volume V_0 initially, with essentially constant initial density ρ_0. Then the mass of the parcel is $\rho_0 V_0$, and is a material invariant. At later times the density is ρ and the volume is $V_0\mathrm{Det}(\mathbf{J})$, so conservation of mass is expressed by

$$\mathrm{Det}\mathbf{J}(\mathbf{a}, t) = \frac{\rho_0}{\rho}. \tag{2.8}$$

If $q \neq 0$ the Lagrangian conservation of mass must be written

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{a}} \rho \mathrm{Det}(\mathbf{J}) = \mathrm{Det}(\mathbf{J}) q(\mathbf{x}(\mathbf{a}, t), t). \tag{2.9}$$

It is easy to get from Eulerian to Lagrangian form using (1.14). Assuming $q = 0$,

$$\frac{D\rho}{Dt} + \rho \,\mathrm{div}\, \mathbf{u} = 0 = \frac{D\rho}{Dt} + \rho \frac{D\mathrm{Det}(\mathbf{J})/Dt}{\mathrm{Det}(\mathbf{J})} = \frac{1}{\mathrm{Det}(\mathbf{J})}\frac{D}{Dt}(\rho \mathrm{Det}(\mathbf{J})) \tag{2.10}$$

and the connection is complete.

Example 2.2: Consider, in one dimension, the unsteady velocity field $u(x,t) = \frac{2xt}{1+t^2}$. We assume no sources of sinks of mass, and set $\rho(x,0) = x$. What is the density field at later times, in both Eulerian and Lagrangian forms? First note that this is a reasonable question, since we have a conservation of mass equation to evolve the density in time. First deriving the Lagrangian coordinates, we have

$$\frac{dx}{dt} = \frac{2xt}{1+t^2}, \quad x(0) = a. \tag{2.11}$$

The solution is $x = a(1+t^2)$. The Jacobian is then $J = 1+t^2$. The equation of conservation of mass in Lagrangian form, given that $\rho_0(a) = a$, is $\rho = a/(1+t^2)$. Since $a = x/(1+t^2)$, the Eulerian form of the density is $\rho = x(1+t^2)^2$. It is easy to check that this last expression satisfies the Eulerian conservation of mass equation in one dimension $\rho_t + (\rho u)_x = 0$.

Example 2.3 Consider the two-dimensional stagnation-point flow $(u,v) = (x,-y)$ with initial density $\rho_0(x,y) = x^2+y^2$ and $q = 0$. The flow is incompressible, so ρ is material. In Lagrangian form, $\rho(a,b,t) = a^2+b^2$. To find ρ as a function of x, y, t, we note that the Lagrangian coordinates of the flow are $(x,y) = (ae^t, be^{-t})$, and so

$$\rho(x,y,t) = (xe^{-t})^2 + (ye^t)^2 = x^2e^{-2t} + y^2e^{2t}. \tag{2.12}$$

The lines of constant density, which are initially circles centered at the origin, are flattened into ellipses by the flow.

2.1.3 Another convection identity

Frequently fluid properties are most conveniently thought of as densities per unit mass rather than per unit volume. If the conservation such a quantity, f say, is to be examined, we will need to consider ρf to get "f per unit volume" and so be able to compute total amount by integration over a volume. Consider then

$$\frac{d}{dt}\int_{\mathcal{S}_t} \rho f dV_{\mathbf{x}} = \int_{\mathcal{S}_t} \Big[\frac{\partial \rho f}{\partial t} + \mathrm{div}(\rho f \mathbf{u})\Big] dV_{\mathbf{x}}. \tag{2.13}$$

We now assume conservation of mass with $q = 0$. From the product rule of differentiation we have $\mathrm{div}(\rho f \mathbf{u}) = f \mathrm{div}(\rho \mathbf{u}) + \rho \mathbf{u} \cdot \nabla f$, and so the integrand splits into a part which vanishes by conservation of mass, and a material derivative of f time the density:

$$\frac{d}{dt}\int_{\mathcal{S}_t} \rho f dV_{\mathbf{x}} = \int_{\mathcal{S}_t} \rho \frac{Df}{Dt} dV_{\mathbf{x}}. \tag{2.14}$$

Thus the effect of the multiplier ρ is to turn the derivative of the integral into an integral of a material derivative.

2.2 Conservation of momentum in an ideal fluid

The *momentum* of a fluid is defined to be $\rho\mathbf{u}$, per unit volume. Newton's second law of motion states that momentum is conserved by a mechanical system of masses if no forces act on the system. We are thus in a position to use (2.14), where the "sources and sinks" of momentum are *forces*.

If $\mathbf{F}(\mathbf{x}, t)$ is the force acting on the fluid, per unit volume, then we have immediately (assuming conservation of mass with $q = 0$),

$$\rho\frac{D\mathbf{u}}{Dt} = \mathbf{F}. \tag{2.15}$$

Since we have seen that $\frac{D\mathbf{u}}{Dt}$ is the fluid acceleration, (2.15) states Newton's Law that mass times acceleration equals force, in both magnitude and direction.

Of course the Lagrangian form of (2.15) is obtained by replacing the acceleration by its Lagrangian counterpart:

$$\rho\frac{\partial^2\mathbf{x}}{\partial t^2}\Big|_{\mathbf{a}} = \mathbf{F}. \tag{2.16}$$

The main issues involved with conservation of momentum are those connected with the forces which are on a parcel of fluid. There are many possible forces to consider: pressure, gravity, viscous, surface tension, electromotive, etc. Each has a physical origin and a mathematical model with a supporting set of observation and analysis. In the present chapter we consider only an *ideal fluid*. The only new fluid variable we will need to introduce is the *pressure*, a scalar function $p(\mathbf{x}, t)$.

In general the force $\mathbf{F}$ appearing in (2.15) is assumed to take the form

$$F_i = f_i + \frac{\partial\sigma_{ij}}{\partial x_j}. \tag{2.17}$$

Here $\mathbf{f}$ is a body force (exerted from the "outside"), and σ is a second-order tensor called the *stress tensor*. Integrated over a region $\mathcal{R}$, the force on the region is

$$\int_{\mathcal{R}}\mathbf{F}dV_{\mathbf{x}} = \int_{\mathcal{R}}\mathbf{f}dV_{\mathbf{x}} + \int_{\partial\mathcal{R}}\sigma\cdot\mathbf{n}dS_{\mathbf{x}}, \tag{2.18}$$

using the divergence theorem. We can thus see that the effect of the stress tensor is to produce a force on the boundary of any fluid parcel, the contribution from an area element to this force being $\sigma_{ij}n_j dS_{\mathbf{x}}$ for an outward normal $\mathbf{n}$. The remaining body force $\mathbf{f}$ will sometimes be taken to be a uniform gravitational field $\mathbf{f} = \rho\mathbf{g}$, where $\mathbf{g}$ = constant. On the surface of the earth gravity acts toward the Earth's center with a strength $g \approx 980\ cm/sec^2$. We also introduce a general force potential Φ, such that $\mathbf{f} = -\rho\nabla\Phi$.

2.2.1 The pressure

An ideal fluid is defined by a stress tensor of the form

$$\sigma_{ij} = -p\delta_{ij} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}, \tag{2.19}$$

where $\delta_{ij} = 1, i = j, = 0$ otherwise. Thus when pressure is positive the force on the surface of a parcel is opposite to the outer normal, as intuition suggests. Note that now

$$\text{div } \sigma = -\nabla p. \tag{2.20}$$

For a compressible fluid the pressure accounts physically for the resistance to compression. But pressure persists as a fundamental source of surface forces for an incompressible fluid, and its physical meaning in the incompressible case is subtle.[1]

An ideal fluid with no mass addition and no body force thus satisfies

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla p = 0, \tag{2.21}$$

together with

$$\frac{D\rho}{Dt} + \rho \text{div } \mathbf{u} = 0. \tag{2.22}$$

This system of equation for an ideal fluid are also often referred to as *Euler's equations*. The term *Euler flow* is also in wide use.

With Euler's system we have $N + 1$ equations for the $N + 2$ unknowns $u_1, \dots, u_N, \rho, p$. Another equation will be needed to complete the system. One possibility is the incompressible assumption div $\mathbf{u} = 0$. A common option is to assume constant density. Then ρ is eliminated as an unknown and the conservation of mass equation is replaced by the incompressibility condition. For gases the missing relation is an equation of state, which brings in the thermodynamic properties of the fluid.

The pressure force as we have defined it above is *isotropic*, in the sense the pressure is the same independently of the orientation of the area element on which it acts. A simple two-dimensional diagram will illustrate why this is so, see figure 2.1. Suppose that the pressure is p_i on the face of length L_i. Equating forces, we have $p_1 L_1 \cos\theta = p_2 L_2, p_1 L_1 \sin\theta = p_3 L_3$. But $L_1 \cos\theta = L_2, L_1 \sin\theta = L_3$, so we see that $p_1 = p_2 = p_3$. So indeed the pressure sensed by a face does not depend upon the orientation of the face.

2.2.2 Lagrangian form of conservation of momentum

The Lagrangian form of the acceleration has been noted above. The momentum equation of an ideal fluid requires that we express ∇p as a Lagrangian variable.

[1]One aspect of the incompressible case should be noted here, namely that the pressure is arbitrary up to an additive constant. Consequently it is only pressure *differences* which matter. This is not the case for a compressible gas.

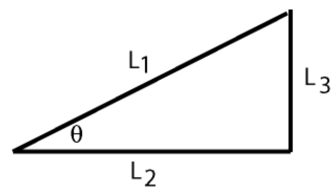

Figure 2.1: Isotropicity of pressure.

That is, if p is to be a function of $\mathbf{a}, t$ then since ∇ here is actually the $\mathbf{x}$ gradient $\nabla_{\mathbf{x}}$, we have $\nabla_{\mathbf{x}} p = \mathbf{J}^{-1}\nabla_{\mathbf{a}} p$. This appearance of the Jacobian is an awkward feature of Lagrangian fluid dynamics, and is one of the reasons that we shall emphasize Eulerian variables in discussing the dynamics of a fluid.

2.2.3 Hydrostatics: the Archimedean principle

Hydrostatics is concerned with fluids at rest ($\mathbf{u} = 0$), usually in the presence of gravity. We consider here only the case of a fluid stratified in one dimension. To fix the coordinates let the z-axis be vertical up, and $\mathbf{g} = -g\mathbf{i}_z$, where g is a positive constant. We suppose that the density is a function of z alone. This allows, for example, a body of water beneath a stratified atmosphere. Let a solid three-dimensional body (any deformation of a sphere for example) be submerged in the fluid. Archimedes principle says that the force exerted by the pressure on the surface of the body is equal to the total weight of the fluid displaced by the body. We want to establish this principle in the case considered.

Now the pressure satisfies $\nabla p = -g\rho(z)\mathbf{i}_z$. The pressure force is given by $\mathbf{F}_{pressure} = -\int p\mathbf{n}dS$ taken over the surface of the body. But this surface pressure is just the same as would be acting on a virtual surface within the fluid, no body present. Using the divergence theorem, we may convert this to an integral over the interior of this surface. Of course, there is no fluid within the body. We are just using the math to evaluate the surface integral. The result is $\mathbf{F}_{pressure} = g\mathbf{i}_z \int \rho dV$. This is a force upward equal to the weight of the displaced fluid, as stated.

2.3 Steady flow of a fluid of constant density

This special case gives us an opportunity to obtain some useful results rather easily in a class of problems of some importance. We shall allow a body force of the form $\mathbf{f} = -\rho\nabla\Phi$, so the momentum equation may be written, after division by the constant density,

$$\mathbf{u} \cdot \nabla\mathbf{u} + \rho^{-1}\nabla p + \nabla\Phi = 0. \tag{2.23}$$

We note now a vector identity which will be useful:

$$\mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \cdot \nabla\mathbf{B} + \mathbf{B} \cdot \nabla\mathbf{A} = \nabla(\mathbf{A} \cdot \mathbf{B}). \tag{2.24}$$

Applying this to $\mathbf{A} = \mathbf{B} = \mathbf{u}$ we have

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{2.25}$$

Using (2.25) in (2.23) we have

$$\nabla(\rho^{-1}p + \Phi + \frac{1}{2}|\mathbf{u}|^2) = \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{2.26}$$

Taking the dot product with $\mathbf{u}$ on both sides we obtain

$$\mathbf{u} \cdot \nabla(\rho^{-1}p + \Phi + \frac{1}{2}|\mathbf{u}|^2) = 0. \tag{2.27}$$

The famous *Bernoulli theorem* for steady flows follows:*In the steady flow of an ideal fluid of constant density the quantity* $H \equiv \rho^{-1}p + \Phi + \frac{1}{2}|\mathbf{u}|^2$, *called the Bernoulli function, is constant on the streamlines of the flow.* The importance of this result is in the relation it gives us between velocity and pressure. Apart from the contribution of Φ, the constancy of H implies that an increase of velocity is accompanied by a decrease of the pressure. This is not an obvious dynamical consequence of the equations of motion, and it is interesting that we have derived it without referring to the solenoidal property of $\mathbf{u}$. Recall that the latter is implied by the constancy of density when there is no mass added or removed. If we make use of the solenoidal property then, using the identity $\nabla \cdot (\mathbf{A}\psi) = \psi\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\phi$ for vector and scaler fields, we see that $\mathbf{u}H$ is also solenoidal, and so the flux of this quantity is conserved in stream tubes. This vector field arises when conservation of *mechanical energy*, relating changes in kinetic energy to the work done by forces, is studied, see problem 2.2.

It is helpful to apply the Bernoulli theorem to flow in a smooth rigid pipe of circular cross section and slowly varying diameter, with $\Phi = 0$. For an ideal (frictionless) fluid we may assume that the velocity is approximately constant over the section, this being reasonable if the slope of the wall of the pipe is small. The velocity may thus be taken as a scalar function $u(x)$. If the section area is $A(x)$, then the conservation of mass (and here, volume) implies that $uA \equiv Q =$ constant, so that $\rho^{-1}p + \frac{Q^2}{2}A^{-2} =$ constant. If we consider a contraction, as in figure 2.2., where the area and velocity go from A_1, u_1 to A_2, u_2, then the fluid speeds up to satisfy $A_1u_1 = A_2u_2 = Q$. To achieve this speedup in steady flow, a force must be acting on the fluid, here a pressure force. Conservation of momentum states the flux of momentum out minus the flux of momentum in must equal the pressure force on the fluid in the pipe between section 1 and section 2. Now $H = p/\rho + \frac{1}{2}(Q/A)^2$ is constant, so (if force is positive to the right) the two ends of the tube give a net pressure force $p_1A_1 - p_2A_2 = \rho Q^2/2(1/A_2 - 1/A_1)$ acting on the fluid. But there is also a pressure force along the curved part of the tube. This is seen to be $\int_{A_1}^{A_2} p dA = -\int_{A_1}^{A_2} \frac{\rho}{2}(Q/A)^2 dA = \rho Q^2/2(1/A_2 - 1/A_1)$. These two contributions are equal in our one-dimensional approximation, and their sum is $\rho Q^2(1/A_2 - 1/A_1)$ But the momentum out minus momentum in is $\rho(A_2u_2^2 - A_1u_1^2) = \rho Q^2(1/A_2 - 1/A_1)$ and is indeed

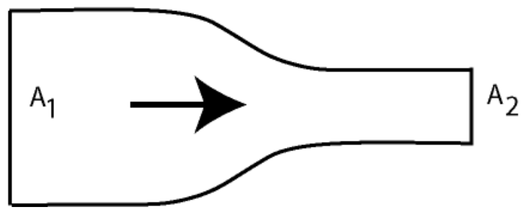

Figure 2.2: Steady flow through a contraction.

equal to the net pressure force. Intuitively then, to achieve the speedup of the fluid necessary to force the fluid through a contraction, and to maintain such a flow as steady in time, it is necessary to supply a larger pressure at station 1 than at station 2. Bernoulli's theorem captures this creation of momentum elegantly, but ultimately the physics comes down to pressure differences accelerating fluid parcels.

2.4 Intrinsic coordinates in steady flow

The one-dimensional analysis just given suggests looking briefly at the relations obtained in an arbitrary steady flow of an ideal fluid using the streamlines a part of the coordinate system. The resulting *intrinsic coordinates* are revealing of the dynamics of fluid parcels. Let $\mathbf{t}$ be the unit tangent vector to an oriented streamline. The we may write $\mathbf{u} = q\mathbf{t}, q = |\mathbf{u}|$. If s is arclength along the streamline, then

$$\frac{\partial \mathbf{u}}{\partial s} = \frac{\partial q}{\partial s}\mathbf{t} + q\frac{\partial \mathbf{t}}{\partial s} = \frac{\partial q}{\partial s}\mathbf{t} + q\kappa\mathbf{n}, \tag{2.28}$$

where $\mathbf{n}$ is the unit normal, κ the streamline curvature, and we have used the first Frenet-Serret formula. Now the operator $\mathbf{u} \cdot \nabla$ is just $q\frac{\partial}{\partial s}$, and so we have from (2.28)

$$\mathbf{u} \cdot \nabla \mathbf{u} = q\frac{\partial q}{\partial s}\mathbf{t} + q^2\kappa\mathbf{n}. \tag{2.29}$$

This shows that the acceleration in steady flow splits into a component along the streamline, determined by the variation of q, and a centripetal acceleration associated with streamline curvature. The equations of motion in intrinsic coordinates (zero body force) are therefore

$$\rho q\frac{\partial q}{\partial s} + \frac{\partial p}{\partial s} = 0, \quad \rho\kappa q^2 + \frac{\partial p}{\partial n} = 0. \tag{2.30}$$

What form does the solenoidal condition take in intrinsic coordinates? We consider this question in two dimensions. We have

$$\nabla \cdot \mathbf{u} = \nabla \cdot (q\mathbf{t}) = \mathbf{t} \cdot \nabla q + q\nabla \cdot \mathbf{t} = \frac{\partial q}{\partial s} + q\nabla \cdot \mathbf{t}. \tag{2.31}$$

Let us introduce an angle θ so that $\mathbf{t}(s) = (\cos\theta(s), \sin\theta(s))$. Then

$$\nabla \cdot \mathbf{t} = -\sin\theta\frac{\partial \theta}{\partial x} + \cos\theta\frac{\partial \theta}{\partial y} = \mathbf{n} \cdot \nabla\theta = \frac{\partial \theta}{\partial n}. \tag{2.32}$$

Since $\kappa = \frac{\partial \theta}{\partial s}$ is the streamline curvature, $\frac{\partial \theta}{\partial n}$, which we write as κ_n, is the curvature of the coordinate lines normal to the streamlines. Thus the solenoidal condition in two dimensions assumes the form

$$\frac{\partial q}{\partial s} + q\kappa_n = 0. \tag{2.33}$$

2.5 Potential flows with constant density

Another important and very large class of fluid flows are the so-called potential flows, defined as flows having a velocity field which is the gradient of a scalar *potential*, usually denoted by ϕ:

$$\mathbf{u} = \nabla\phi. \tag{2.34}$$

For simplicity we consider here only the case of constant density, but allow a body force $-\rho\nabla\Phi$ and permit the flow to be unsteady. Since we now also have that $\mathbf{u}$ is solenoidal, it follows that

$$\nabla \cdot \nabla\phi = \nabla^2\phi = 0. \tag{2.35}$$

Thus the velocity field is determined by solving Laplace's equation (2.35)

The momentum equation has not yet been needed, but it necessary in order to determine the pressure, given $\mathbf{u}$. The momentum equation is

$$\mathbf{u}_t + \nabla(\frac{1}{2}|\mathbf{u}|^2 + p/\rho + \Phi) = \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{2.36}$$

Since $\mathbf{u} = \nabla\phi$ we now have $\nabla \times \mathbf{u} = 0$ and therefore

$$\nabla(\phi_t + \frac{1}{2}|\nabla\phi|^2 + p/\rho + \Phi) = 0, \tag{2.37}$$

or

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + p/\rho + \Phi = h(t). \tag{2.38}$$

The arbitrary function $h(t)$ may in fact be set equal to zero; otherwise we can replace ϕ by $\phi - \int h dt$ without affecting $\mathbf{u}$. We see that (2.38) is another "Bernoulli constant", this time applicable to any connected region of space where the potential flow is defined. It allows us to compute the pressure in an unsteady potential flow, see problem 2.6.

2.6 Boundary conditions on an ideal fluid

As we have noted, a main physical property of real fluid which is not present for an ideal fluid is a viscosity. The ideal fluid is"slippery", in the following sense. Suppose that adjacent to a solid wall the pressure varies along the wall. The only force a fluid parcel can experience is a pressure force associated with the pressure gradient. If the gradient at the wall is tangent to the wall, fluid will be accelerated and there will have to be a tangential component of velocity *at the wall.* This suggests that we cannot place any restriction on the tangential component of velocity at a rigid fixed boundary of the fluid.

On the other hand, by a rigid fixed wall we mean that fluid is unable to penetrate the wall, and so we will have to impose the condition $\mathbf{n}\cdot\mathbf{u} = u_n = 0$ on the wall. There is a subtlety here connected with our continuum approximation. It might be though that the fluid cannot penetrate *into* a rigid wall, but could it not be possible for the fluid to tear off the wall, forming a free interface next to an empty cavity? to see that this cannot be the case for smooth pressure fields, consider the reversed stagnation-point flow $(u, v) = (-x, y)$. On the upper y-axis we have a Bernoulli function $p/\rho + \frac{1}{2}y^2$. The gradient of pressure along this line is indeed accelerating the fluid away from the wall, but the fluid remains at rest at $x = y = 0$. We cannot really contemplate a pressure force on a particle, which might cause the particle to leave the wall, only on a parcel. In fact in this example fluid parcels near the y-axis are being compressed in the x-direction and stretched in the y-direction.

Thus, the appropriate boundary condition at a fixed rigid wall adjacent to an ideal fluid is

$$u_n = 0 \quad \text{on the wall.} \tag{2.39}$$

For a potential flow, this becomes

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on the wall.} \tag{2.40}$$

We shall find that these conditions at a rigid wall for an ideal fluid are sufficient to (usually uniquely) determine fluid flows in problems of practical importance.

Another way to express the appropriate boundary condition on a ideal fluid at a rigid wall is that *fluid particles on a wall stay on the wall.* This alternative is attractive because it is also true of a *moving* rigid wall, where the velocity component normal to the wall need not vanish at the wall. So what is the appropriate condition on a moving wall? To obtain this it is convenient to define the surface as a function of time by the equation $\Sigma(\mathbf{x}, t) = 0$. For a particle at position $\mathbf{x}_p(t)$ to be on the surface means that $\Sigma(\mathbf{x}_p(t), t) = 0$. Differentiating this expression with respect to time we obtain

$$\left.\frac{\partial\Sigma}{\partial t}\right|_{\mathbf{x}} + \mathbf{u}\cdot\nabla\Sigma = 0. \tag{2.41}$$

For example, let a rigid cylinder of radius a move in the x-direction with velocity U. Then $\Sigma = (x - Ut)^2 + y^2 - a^2$, and (2.41) becomes $-2U(x - Ut) + 2(x -$

$Ut)u + 2yv = 0$ Evaluating this on the surface of the cylinder, we get

$$u \cos\theta + v \sin\theta = U \cos\theta = u_n. \tag{2.42}$$

We remark that the same reasoning can be applied to the moving *interface* between two fluids. This interface may also be regarded as consisting of fluid particles that remain on the interface. We refer to this generalized boundary condition at a moving surface as a *surface condition*.

Finally, as part of this first look at the boundary condition of fluid dynamics, we should note that for unsteady fluid flows we will sometimes need to prescribe *initial conditions*, insuring that the fluid equations may be used to carry the solution forward in time.

Example 2.4: We consider an example of potential flow past a body in two dimensions, constant density, no body force. The body is the circular cylinder $r = a$, and the fluid " at infinity" has fixed velocity $(U, 0)$ In two dimensional polar coordinates, Laplace's equation has solutions of the form $\ln r, (r^n, r^{-n})(\cos\theta, \sin\theta),\ n = 1, 2, \ldots$. The potential $Ur\cos\theta = Ux$ has the correct behavior at infinity, and so we need a decaying solution which will insure the boundary condition $\frac{\partial\phi}{\partial r} = 0$ when $r = a$. The correct choice is clearly a multiple of $r^{-1}\cos\theta$ and we obtain

$$\phi = U\cos\theta(r + a^2/r) \tag{2.43}$$

Note that $U\cos a^2/r$ is the potential of a flow seen by an observer at rest relative to the fluid at infinity, when the cylinder moves relative to the fluid with a velocity $(-U, 0)$. We see that indeed this potential satisfies $\frac{\partial\phi}{\partial r}\big|_{r=a} = -U\cos\theta$ as required by (2.42). Streamlines both inside and outside the cylinder are shown in figure 2.3.

We have found a solution representing the desired flow, but is the solution unique? Perhaps surprisingly, the answer is no. The reason, associated with the fluid region being non-simply connected, will be discussed in chapter 4.

Example 2.5 An interesting case of unsteady potential flow occurs with deep water waves (constant density). The fluid at rest is a liquid in the domain $z < 0$ of R^3. Gravity acts downward so $\Phi = -gz$. The space above is taken as having no density and a uniform pressure p_0. If the water is disturbed, waves can form on the surface, which we will assume to be described by a function $z = Z(x, y, t)$ (no breaking of waves). Under appropriate initial conditions it turns out that we may assume the liquid velocity to be a potential flow. Thus our mathematical problem is to solve Laplace's equation in $z < Z(x, y, t)$ with a surface condition on ϕ and a pressure condition $p_{z=Z} = p_0$. For the latter we can use the Bernoulli theorem for unsteady potential flows, to obtain

$$p_0/\rho = \Big[-\phi_t - \frac{1}{2}|\nabla\phi|^2 + gz \Big]_{z=Z}. \tag{2.44}$$

The surface condition is $\frac{D}{Dt}(z - Z(x, y, t)) = 0$ or

$$\Big[z - Z_t - uZ_x - vZ_y \Big]_{z=Z} = 0. \tag{2.45}$$

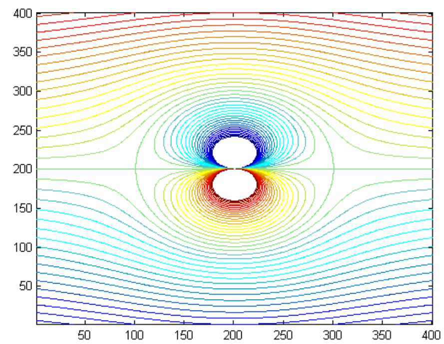

Figure 2.3: Potential flow past a circular cylinder.

The object is to find $\phi(x, y, z, t), Z(x, y, t)$, given e.g. that the water is initially at rest and that the fluid surface is at an initial elevation $z = Z_0(x, y)$. We will consider water waves in more detail in Chapter 9.

Problem set 2

1. For potential flow over a circular cylinder as discussed in class, with pressure equal to the constant p_∞ at infinity , find the static pressure on surface of the cylinder as a function of angle from the front stagnation point. (Use Bernoulli's theorem.) Evaluate the drag force (the force in the direction of the flow at infinite which acts on the cylinder), by integrating the pressure around the boundary. Verify that the drag force vanishes. This is an instance of *D'Alembert's paradox*, the vanishing of drag of bodies in steady potential flow.

2. For an ideal inviscid fluid of constant density, no gravity, the conservation of mechanical energy is studied by evaluating the time derivative of total kinetic energy in the form

$$\frac{d}{dt}\int_D \frac{1}{2}\rho|\mathbf{u}|^2 dV = \int_{\partial D} \mathbf{F}\cdot\mathbf{n}dS.$$

Here D is an arbitrary fixed domain with smooth boundary ∂D. What is the vector $\mathbf{F}$? Interpret the terms of $\mathbf{F}$ physically.

3. An open rectangular vessel of water is allowed to slide freely down a smooth frictionless plane inclined at an angle α to the horizontal, in a uniform vertical gravitational field of strength g. Find the inclination to the horizontal of the free surface of the water, given that it is a surface of constant pressure. We assume the fluid is at rest relative to an observer riding on the vessel. (Consider the acceleration of the fluid particles in the water and balance this against the gradient of pressure.)

4. Water (constant density) is to be pumped up a hill (gravity $= (0, 0, -g)$) through a pipe which tapers from an area A_1 at the low point to the smaller area A_2 at a point a vertical distance L higher. What is the pressure p_1 at the bottom, needed to pump at a volume rate Q if the pressure at the top is the atmospheric value p_0? (Express in terms of the given quantities. Assuming inviscid steady flow, use Bernoulli's theorem with gravity and conservation of mass. Assume that the flow velocity is uniform across the tube in computing fluid flux and pressure.)

5. For a *barotropic fluid*, pressure is a function of density alone, $p = p(\rho)$. In this case derive the appropriate form of Bernoulli's theorem for steady flow without gravity. If $p = k\rho^\gamma$ where γ, k are positive constants, show that $q^2 + \frac{2\gamma}{\gamma-1}\frac{p}{\rho}$ is constant on a streamline, where $q = |\mathbf{u}|$ is the speed.

6. Water fills a truncated cone as shown in the figure. Gravity acts down (the direction $-z$). The pressure at the top surface, of area A_2 is zero. The height of the container is H. At $t = 0$ the bottom, of area $A_1 < A_2$, is abruptly removed and the water begins to fall out. Note that at time $t = 0+$ the pressure at the bottom surface is also zero. The water has not moved but the acceleration is non-zero. We may assume the resulting motion is a potential flow. Thus the potential

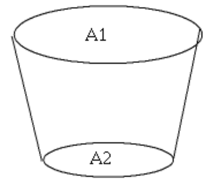

Figure 2.4: Truncated cone of fluid

$\phi(z, r, t)$ in cylindrical polars has the Taylor series $\phi(r, z, t) = t\Phi(r, z) + O(t^2)$, so $d\phi/dt = \Phi(r, z) + O(t)$. Using these facts, set up a mathematical problem for determining the pressure on the inside surface of cone at $t = 0+$. You should specify all boundary conditions. You do not have to solve the resulting problem, but can you guess what the surfaces Φ =constant would look like qualitatively? What is the force felt at $t = 0+$ by someone holding the cone, in the limits $A_1 \to 0$ and $A_1 \to A_2$?

Chapter 3

Vorticity

We have already encountered the vorticity field in the identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \frac{1}{2} |\mathbf{u}|^2 - \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{3.1}$$

The vorticity field $\boldsymbol{\omega}(\mathbf{x}, t)$ is defined from the velocity field by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}. \tag{3.2}$$

A potential flow is a flow with zero vorticity. The term *irrotational flow* is widely used. According to (3.1) the contribution to the acceleration coming from the gradient of velocity can be split into two components, one having a potential $\frac{1}{2}|\mathbf{u}|^2$, the other given as a cross product orthogonal to both the velocity and the vorticity. The latter component in older works in fluid dynamics has been called the *vortex force*.

We remark that, in analogy with stream lines, we shall refer to the flow lines of the vorticity field, i.e. the integral curves of the system

$$\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}, \tag{3.3}$$

as (instantaneous) *vortex lines*. Similarly, in analogy with a stream tube in three dimensions, we will refer to a bundle of vortex lines a *vortex tube*.

This straightforward definition of the vorticity field gives little insight into its importance, either physically and theoretically. This chapter will be devoted to examining the vorticity field from a variety of viewpoints.

3.1 Local analysis of the velocity field

The first thing to be noted is that vorticity is fundamentally an Eulerian property since it involves spatial derivatives of the Eulerian velocity field. In a sense the analytical structure of the flow is being expanded to include the first derivatives of the velocity field. Suppose we expand the velocity field in a Taylor series about the fixed point $\mathbf{x}$:

$$u_i(\mathbf{x}+\mathbf{y},t) = u_i(\mathbf{x},t) + y_j \frac{\partial u_i}{\partial x_j}(\mathbf{x},t) + O(|\mathbf{y}|^2). \tag{3.4}$$

We can make the division

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big] + \frac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big]. \tag{3.5}$$

The term first term on the right, $\frac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big]$, is often denoted by e_{ij} and is the *rate-of-strain tensor* of the fluid. Here it will play a basic role when viscous stresses are considered (Chapter 5). The second term, $\frac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big]$, can be seen to be, in three dimensions, the matrix

$$\mathbf{\Omega} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{3.6}$$

Example 3.1: In two dimensions, since u, v depend only on x, y, only one component of the vorticity is non-zero, $\omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. This is usually written simply as the scalar ω. Consider the two-dimensional flow $(u,v) = (y,0)$. In this case

$$\mathbf{e} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{\Omega} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.7}$$

and $\omega = -1$. This is a simple "shear flow" with horizontal particle paths. Both $\mathbf{e}$ and $\mathbf{\Omega}$ are non-vanishing.

Example 3.2: Consider the flow $(u,v) = (-y,x)$. This is a simple solid-body rotation in the anti-clockwise sense. The vorticity is $\omega = 2$, and $\mathbf{e} = 0$.

These examples are a bit atypical because the vorticity is constant, but they emphasize that a close association of the vorticity with fluid rotation, a connection suggested by the skew-symmetric form of $\mathbf{\Omega}$, can be misleading.

Vorticity is a *point* property, but can only be defined by the limit operations implicit in the needed derivatives. So it is impossible to attach a physical meaning to "the vorticity of a particle". We *can* truncate (3.4) and consider the Lagrangian paths of fluid particles near $\mathbf{x}$. Since $\mathbf{e}$ is real symmetric, it may be diagonalized by a rotation to principal axes. Let the eigenvalues along the diagonal be $\lambda_i, i = 1,2,3$. We the may assume our coordinate system is such that $\mathbf{e}$ is the diagonal matrix $\mathbf{D}(\mathbf{x})$. Then the Lagrangian coordinates of the perturbed path $\mathbf{y}$ satisfies

$$\mathbf{y}_t = \mathbf{D}(\mathbf{x})\mathbf{y} + \frac{1}{2}\boldsymbol{\omega}(\mathbf{x}) \times \mathbf{y}. \tag{3.8}$$

These equations couple together the rotation associated with the vorticity at $\mathbf{x}$ with the straining field described by the first term. Note that the angular velocity associated with second term is $\frac{1}{2}\boldsymbol{\omega}$. The statement " vorticity at $\mathbf{x}$ equals twice the angular velocity of the fluid at $\mathbf{x}$" is often heard. But this statement in fact makes no sense, since an angular velocity cannot be attributed to a point. *Given the velocity field of a fluid, one can determine the effects of vorticity on the fluid only on a small open set, i.e. a fluid parcel.*

On the other hand it is true that when vorticity is sufficiently large there is sensible rotation observed in the fluid, and it *is* true that when one sees "rotation" in the fluid, then vorticity is present. In a sense this is the key to understanding its role, since it forces a definition of "rotation" in a fluid.

3.2 Circulation

Let C be a simple, smooth, oriented closed contour which is a deformation of a circle, hence the boundary of an oriented surface S. Now Stokes' theorem applied to the velocity field states that

$$\int_C \mathbf{u} \cdot d\mathbf{x} = \int_S \mathbf{n} \cdot (\nabla \times \mathbf{u}) dS, \tag{3.9}$$

where the direction of the normal $\mathbf{n}$ to S is chosen from the orientation of C by the "right-hand rule". We can interpret the right-hand side of (3.9) as the *flux of vorticity through S.* So it must be that the left-hand side is an expression of the effect of vorticity *on the velocity field*. We thus define *the fluid circulation of the velocity field* $\mathbf{u}$ *on the contour* C by

$$\Gamma_C = \oint_C \mathbf{u} \cdot d\mathbf{x}. \tag{3.10}$$

The circulation is going to be our measure of the rotation of the fluid.

The key "point" is that is that circulation is defined *globally*, *not* at a point. We need to consider an open set containing S in order to make this definition.

Example 3.3: Potential flows have the property that circulation vanishes on any closed contour, as long as $\mathbf{u}$ is well-behaved in an open set containing S. This is an obvious property of an irrotational flow.

Example 3.4 In two dimensions, the flow $(u, v) = \frac{1}{2\pi}(-y/r^2, x/r^2)$ is a point vortex. If C is a simple closed curve encircling the origin, then Γ_C is equal to the circulation on a circle centered at the origin, by independence of path since (u, v) is irrotational everywhere except at the origin. The circulation on a circle, taken counter-clockwise, is found to be unity. Indeed in polar form the velocity is given by $u_r = 0, u_\theta = \frac{1}{2\pi r}$. The circulation on the circle of radius r is thus $\frac{2\pi r}{2\pi r} = 1$. This flow is called the *point vortex of unit strength.*

3.3 Kelvin's theorem for a barotropic fluid

In chapters 12-14 we will be taking up the dynamics of general compressible fluids. The intervening discussion will deal with only a restricted class of compressible flows, the *barotropic fluids.* A barotropic fluid is defined by specifying pressure as a given function of the density, $p(\rho)$. This reduces the dependent variables of an ideal fluid to $\mathbf{u}, \rho$ and so the system of momentum and mass equations is closed.

Theorem 1 *(Kelvin's theorem) Let $C(t)$ be a simple close* material *curve in an ideal fluid with body force $-\rho\nabla\Phi$. Then, if either (i) ρ = constant, or (ii) the fluid is barotropic, then the circulation $\Gamma_{C(t)}$ of $\mathbf{u}$ on C is invariant under the flow:*

$$\frac{d}{dt}\Gamma_{C(t)} = 0. \tag{3.11}$$

To prove this consider a parametrization $\mathbf{x}(\alpha, t)$ of $C(t)$, $0 \le \alpha \le A$. Then

$$\frac{d}{dt}\oint_C \mathbf{u}\cdot d\mathbf{x} = \frac{d}{dt}\int_0^A \mathbf{u}\cdot\frac{\partial\mathbf{x}}{\partial\alpha}d\alpha = \int_0^A \Big[\frac{D\mathbf{u}}{Dt}\cdot\frac{\partial\mathbf{x}}{\partial\alpha} + \mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial\alpha}\Big]d\alpha. \tag{3.12}$$

Making use of the momentum equation $\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$ we have

$$\frac{d\Gamma_C}{dt} = \int_0^A \Big[-(\frac{1}{\rho}\nabla p + \nabla\Phi)\cdot\frac{\partial\mathbf{x}}{\partial\alpha} + \mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial\alpha}\Big]d\alpha, \tag{3.13}$$

This becomes

$$\frac{d\Gamma_C}{dt} = \oint_C \Big[\frac{-dp}{\rho} + d(\frac{1}{2}|\mathbf{u}|^2 - \Phi)\Big]. \tag{3.14}$$

Now if ρ is a constant, or if the fluid is barotropic, the integrand may be written as perfect differential (in the barotropic case a differential of $-\int \rho^{-1}\frac{dp}{d\rho}d\rho + \frac{1}{2}|\mathbf{u}|^2 - \Phi$). Since all variables are assumed single-valued, the integral vanishes and the theorem is proved.

Kelvin's theorem is a cornerstone of ideal fluid theory since it expresses a global property of vorticity, namely the flux through a surface, as an invariant of the flow. We shall see that it is very useful in understanding the kinematics of vorticity.

3.4 The vorticity equation

In the present section we again assume that either ρ = constant, or else the fluid is barotropic.

In either case it is of interest to consider an equation for vorticity, which can be obtained by taking the curl of

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0. \tag{3.15}$$

Under the conditions stated, this will give

$$\nabla\times\frac{D\mathbf{u}}{Dt} = 0. \tag{3.16}$$

Recalling $\mathbf{u}\cdot\nabla\mathbf{u} = \nabla\frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\times\boldsymbol{\omega}$, we use the vector identity

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\mathbf{A} - \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A}. \tag{3.17}$$

For the case of constant density and no mass addition, both $\nabla\cdot\mathbf{u}$ and $\nabla\cdot\boldsymbol{\omega}$ vanish, with the result

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\mathbf{u}. \tag{3.18}$$

For a barotropic fluid, we need to bring in conservation of mass to evaluate $\nabla\cdot\mathbf{u} = -\rho^{-1}D\rho/Dt$. We then get in place of (3.18)

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \frac{\boldsymbol{\omega}}{\rho}\frac{D\rho}{Dt}. \tag{3.19}$$

This can be rewritten as

$$\frac{D(\frac{\boldsymbol{\omega}}{\rho})}{Dt} = \frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\mathbf{u}. \tag{3.20}$$

Now we want to compare (3.18) and (3.20) with (1.23), and observe that $\boldsymbol{\omega}$ in the first case and $\boldsymbol{\omega}/\rho$ is the second is a *material vector field* as we defined it in chapter 1. This is a deep and remarkable property of the vorticity field, which gives it its importance in fluid mechanics. It tells us, for example, that vorticity magnitude can be increased if two nearby fluid particles lying on the same vortex line move apart.

Example 3.5 In two dimensions $\boldsymbol{\omega}\cdot\nabla\mathbf{u} = 0$ and so the vorticity ω satisfies

$$\frac{D\omega}{Dt} = 0, \tag{3.21}$$

i.e. in two dimensions, for the cases studied here, vorticity is a scalar material invariant, whose value is always the same on a given fluid parcel.

In three dimensions the term $\boldsymbol{\omega}\cdot\nabla\mathbf{u}$ is sometimes called the *vortex stretching term*. Its existence make two and three-dimensional vorticity behaviors entirely different.

There is a Lagrangian form of the vorticity equation, due to Gauss. We can obtain it here by recalling that $v_i(\mathbf{a},t) = J_{ij}(\mathbf{a},t)V_i(\mathbf{a})$ defines a material vector field. Let us assume that, given the initial velocity and therefore initial vorticity fields, vorticity may be solved for uniquely at some function time t using Euler's equations. Then, any material vector field assuming the assigned initial values for vorticity must be the unique vorticity field $\boldsymbol{\omega}$. However, if the initial vorticity is $\boldsymbol{\omega}_0(\mathbf{x})$, then a material vector field which takes on these initial values is $\mathbf{J}(\mathbf{a},t)\cdot\boldsymbol{\omega}_0(\mathbf{a})$. By uniqueness, we must have

$$\omega_i(\mathbf{a},t) = J_{ij}(\mathbf{a},t)\omega_{0j}. \tag{3.22}$$

in the constant density case. For the barotropic case, given initial density $\rho_0(\mathbf{x})$, the corresponding equation is

$$\rho^{-1}\omega_i(\mathbf{a},t) = \rho_0^{-1}(\mathbf{a})J_{ij}(\mathbf{a},t)\omega_{0j}. \tag{3.23}$$

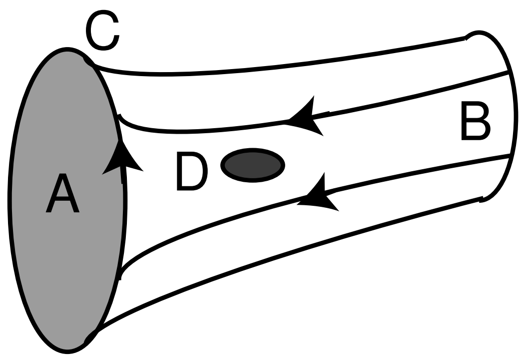

Figure 3.1: A segment of an oriented vortex tube.

This is Cauchy's "solution" of the vorticity equation . Of course nothing has been solved, only represented in terms of the unknown Jacobian. It is however a revealing relation which directly ties the changes in vorticity to the deformation experienced by a fluid parcel.

3.5 Helmholtz' Laws

In dicussing the behavior of vorticity in a fluid flow we will want to consider as our basic element a section of a vortex tube as shown in figure 3.1. Recall that a vortex tube is a bundle of vortex lines, each of the lines being the instantaneous flow lines of of the vorticity field.

In the mid-nineteenth century Helmholtz laid the foundations for the mechanics of vorticity. His conclusions can be summarized by the following three laws:

- Fluid parcels free of vorticity stay free of vorticity.
- Vortex lines are material lines.
- The strength of a vortex tube, to be defined below, is an invariant of the motion.

We have seen that the vorticity field, or the field divided by density in the barotropic case, is a material vector field. The vortex lines are the same in each case if $\boldsymbol{\omega}$ is the same. Hence particles on a particular vortex line at one time, remain on a line at a later time, and so the line is itself material. Thus the tube segment in figure 3.1 is bounded laterally by a surface of vortex lines. The small patch D in the surface thus carries no flux of vorticity. The bounding contour of this patch is a material curve, and by Kelvin's theorem the circulation on the contour is a material invariant. Since this circulation is initially zero by the absence of flux of vorticity through the patch, it will remain zero. Consequently the lateral boundary of a vortex tube remains a boundary of the tube.

In follows from the solenoidal property of vorticity and the divergence theorem that the flux of vorticity through the end surface A, must equal that through the end surface B. This flux is a property of a vortex tube, called the *vortex tube strength*. Note that this is independent of the compressibility or incompressibility of the fluid. The tube strength expresses simply a property of a solenoidal vector field.

To establish the third law of Helmholtz we must show that this strength is a material invariant. But this follows immediately from Kelvin's theorem, since the circulation on the contour C is a material invariant. This circulation, for the orientation of the contour shown in the figure, is equal to the vortex tube strength by Stokes' theorem, and we are done.

The first law is also established using Kelvin's theorem. Suppose that a flow is initially irrotational but at some time a fluid parcetionl is found where vorticity is non-zero. A small closed contour can then be found with non-vanishing circula is non-zero, by Kelvin's theorem. This contradicts the irrotationality of the initial flow.

Using these laws we may see how changes in the shape of a fluid parcel can change the magnitude of vorticity. In figure 3.2 we show a segment of small vortex tube which has changes under the flow from have length $L_1$and section area A_1, to new values A_2, L_2. If the density is constant, volume is conserved, $A_1L_1 = A_2L_2$. If the vorticity magnitudes are ω_1, ω_2, then invariance of the tube strength implies $\omega_1 A_1 = \omega_2 A_2$. Comparing these expressions, $\omega_2/\omega_1 = L_2/L_1$. Consequently, *for an ideal fluid of constant density the vorticity is proportional to vortex line length*. We understand here that by line length we are referring to the distance between to nearby fluid particles on the same vortex line. Thus the growth or decay of vorticity in ideal fluid flow is intimately connected to the stretching properties of the Lagrangian map.[1] Fluid tubulence is observed to contain small domains of very large vorticity, presumably created by this stretching.

For a compressible fluid the volume of te tube need not be invariant, but mass is conserved. Thus we have, introducing the initaial and final densities ρ_1, ρ_2,

$$\rho_1 A_1 L_1 = \rho_1 A_2 L_2, \quad \omega_1 A_1 = \omega_2 A_2. \tag{3.24}$$

It follows that

$$\frac{\omega_2/\rho_2}{\omega_1/\rho_1} = L_2/L_1. \tag{3.25}$$

Thus we see that it is the magnitude of the material field, whether $|\boldsymbol{\omega}|$ or $\rho^{-1}|\boldsymbol{\omega}|$, which is proportional to line length. Notice that in a compressible fluid vorticity may be increased by compressing a tube while holding the length fixed, so as to increase the density.

[1] This make chaotic flow, with positive Liapunov exponents, of great interest in amplifying vorticity.

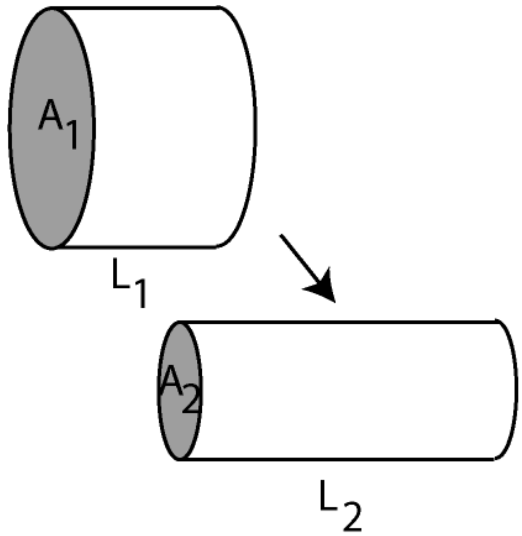

Figure 3.2: Deformation of a vortex tube under a flow.

3.6 The velocity field created by a given vorticity field

Suppose that in R^3 the vorticity field is non-zero in some region and vanishes at infinity. What is the velocity field or fields is created by this vorticity? It is clear that given a vorticity field $\boldsymbol{\omega}$, and a vector field $\mathbf{u}$ such that $\nabla \times \mathbf{u} = \boldsymbol{\omega}$, another vector field with the same property is given by $\mathbf{v} = \mathbf{u} + \nabla\phi$ for some scalar field ϕ, uniqueness is an issue. However, under appropriate conditions a unique construction is possible.

Theorem 2 *Let the given vorticity field be smooth and vanish strongly at infinity, e.g. for some* $R > 0$

$$|\boldsymbol{\omega}| \leq Cr^{-N}, \quad r > R, r = \sqrt{x^2 + y^2 + z^2} \tag{3.26}$$

Then there exists a unique solenoidal vector field $\mathbf{u}$ *such that* $\nabla \times \mathbf{u} = \boldsymbol{\omega}$ *and* $\lim_{r\to\infty} |\mathbf{u}| = 0$. *This vector field is given by*

$$\mathbf{u} = \frac{1}{4\pi} \int_{R^3} \frac{(\mathbf{y} - \mathbf{x}) \times \boldsymbol{\omega}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} dV_{\mathbf{y}}. \tag{3.27}$$

To prove this consider the vector field $\mathbf{v}$ defined by

$$\mathbf{v} = \frac{1}{4\pi} \int_{R^3} \frac{\boldsymbol{\omega}}{|\mathbf{x} - \mathbf{y}|} dV_{\mathbf{y}}. \tag{3.28}$$

This field exists and given (3.26) and can be differentiated if $\boldsymbol{\omega}$ is a smooth function. Let $\mathbf{u} = \nabla \times \mathbf{v}$. Now we have the vector identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}. \tag{3.29}$$

The right-hand side of (3.28) is the unique solution of the vector equation $\nabla^2 \mathbf{v} = \boldsymbol{\omega}$ which vanishes at infinity. Also

$$\begin{aligned} \operatorname{div} \int_{R^3} \boldsymbol{\omega} \cdot \nabla_{\mathbf{x}} \frac{1}{|\mathbf{x}-\mathbf{y}|} dV_{\mathbf{y}} &= -\int_{R^3} \boldsymbol{\omega} \cdot \nabla_{\mathbf{y}} \frac{1}{|\mathbf{x}-\mathbf{y}|} dV_{\mathbf{y}} \\ &= -\int_{R^3} \nabla_{\mathbf{y}} \cdot \Big[\frac{\boldsymbol{\omega}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \Big] dV_{\mathbf{y}} = 0 \end{aligned} \tag{3.30}$$

by the divergence theorem and the fact that the integral of $|\boldsymbol{\omega}|$ over $r = R$ be bounded in R in (3.26) holds. Thus $\mathbf{u}$ as defined by (3.27) satisfies $\nabla \times \mathbf{u} = \boldsymbol{\omega}$. as required. Also, this vector field is solenoidal since it is the curl of $\mathbf{v}$, and vanishes as $|\mathbf{x}| \to \infty$. And it is unique. Indeed if $\mathbf{u}'$ is another vector field with the same properties, then $\nabla \times (\mathbf{u} - \mathbf{u}') = 0$ and so $\mathbf{u} - \mathbf{u}' = \nabla \phi$ for some scale field whose gradient vanishes at infinity. But by the solenoidal property of $\mathbf{u}, \mathbf{u}'$ we see that $\nabla^2 \phi = 0$, and this implies $\phi = \mathit{constant}$, giving the uniqueness of $\mathbf{u}$.

For compressible flows a general velocity field $\mathbf{w}$ with vorticity $\boldsymbol{\omega}$ will have the form $\mathbf{w} = \mathbf{u} + \nabla \phi$ where $\mathbf{u}$ is given by (3.27) and ϕ is an arbitrary scalar field.

The kernel

$$\frac{1}{4\pi} \frac{(\mathbf{y}-\mathbf{x}) \times (\cdot)}{|\mathbf{x}-\mathbf{y}|^3} \tag{3.31}$$

is interesting in the insight it gives into the creation of velocity as a cross product operation. The velocity induced by a small segment of vortex tube is orthogonal to both the direction of the tube and the vector joining the observation point to the vortex tube segment. A similar law relates magnetic field created by an electric cucrrent, where it is know as the *Biot-Savart law*.

3.7 Some examples of vortical flows

We end this chapter with a few examples of ideal fluid flows with non-zero vorticity.

3.7.1 Rankine's combined vortex

This old example is an interesting use of a vortical flow to model a "bath tub vortex", before the depression of the surface of the fluid develops a "hole". It will also give us an example of a flow with a free surface. The fluid is a liquid of constant density ρ with a free surface given by $z = Z(r)$ in cylindrical polar coordinates, see figure 3.3. The pressure above the free surface is the constant p_0. The body force is gravitational, $\mathbf{f} = -g\mathbf{i}_z$. The vorticity is a solid-body rotation in a vertical tube bounded by $r = a, z < Z$. The only nonzero velocity component it the θ-component u_θ.

In $r > a, z < Z$ Euler's equations will be solved by the field of a two-dimensional point vortex (actually a *line vortex*). This will be matched with a rigid rotation for $r < a$ so that velocity is continuous:

$$u_\theta = \begin{cases} \Omega a^2/r, & \text{when } r \geq a, \\ \Omega r, & \text{when } r < a. \end{cases} \tag{3.32}$$

Here Ω is the angular velocity of the core vortex. Now in the exterior region $r > a$ the flow is irrotational and so we have by the Bernoulli theorem for irrotational flows

$$\frac{p_{ext}}{\rho} = \frac{p_0}{\rho} - \frac{1}{2}\Omega^2 a^4 r^{-2} - gz, \tag{3.33}$$

for $z < Z$, where we have taken $Z = 0$ at $r = \infty$. The free surface is thus given for $r > a$ by

$$Z = -\frac{\Omega^2 a^4}{gr^2}. \tag{3.34}$$

Inside the vortex core, the equations reduce to

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{u_\theta^2}{r} = \Omega^2 r, \quad \frac{1}{\rho}\frac{\partial p}{\partial z} = g. \tag{3.35}$$

Thus

$$\frac{1}{2}\Omega^2 r^2 - gz + C \equiv \frac{p_{core}}{\rho}, \quad r < a, z < Z. \tag{3.36}$$

On the cylinder $r = a, z < Z$ we require that the $p_{core} = p_{ext}$, so

$$\frac{1}{2}\Omega^2 a^2 - gz + C = \frac{p_0}{\rho} - gz - \frac{1}{2}\Omega^2 a^2. \tag{3.37}$$

Therefore the constant C is given by

$$C = \frac{p_0}{\rho} - \Omega^2 a^2, \tag{3.38}$$

and

$$\frac{p_{core}}{\rho} = \frac{p_0}{\rho} - \Omega^2 a^2\Big(1 - \frac{r^2}{2a^2}\Big) - gz. \tag{3.39}$$

The free surface is then given by

$$Z = \begin{cases} -\frac{a^4\Omega^2}{2gr^2}, & \text{when } r \geq a, \\ \frac{\Omega^2 a^2}{g}\Big(\frac{r^2}{2a^2} - 1\Big), & \text{when } r < a./\text{cr} \end{cases} \tag{3.40}$$

We have used the adjective "combined" to emphasize that this vortex flow is an example of a solution of the equations of motions which is not smooth, since du_θ/dr is not continuous at $r = a, z < Z$. Since all other components

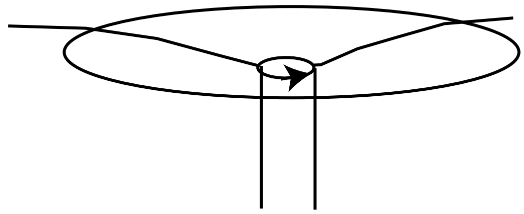

Figure 3.3: Rankine's combined vortex

of velocity are zero and the pressure is the only variable with a z-dependence, the equation are in fact satisfied everywhere. In a real, viscous fluid, if the ideal flow was taken as an initial condition, the irregularity at $r = a$ would be immediately smoothed out by viscous stresses. The ideal fluid solution would nonetheless be a good representation of the flow for some time, until the vortex core is substantially affected by the viscosity.

3.7.2 Steady propagation of a vortex dipole

We consider steady two-dimensional flow of an ideal fluid of constant density, no body force. Since then $\mathbf{u} \cdot \nabla\omega = 0$, introducing the stream function ψ, $(u, v) = (\psi_y, -\psi_x)$, we have

$$\psi_y(\nabla^2\psi)_x - \psi_x(\nabla^2\psi)_y = 0. \tag{3.41}$$

Consequently contours of constant ψ and of constant ω must agree, and so

$$\nabla^2\psi = f(\psi). \tag{3.42}$$

where the function f is arbitrary. We will look for solutions of the simplest kind, by choosing $f = -k^2\psi$, where k is a constant. Using polar coordinates, we look for solutions of the equation $\nabla^2\psi + k^2\psi$ in the disc $r < a$, which can match with the velocity in $r > a$ that is the same as irrotational flow past a circular body of radius a. That potential flow is easily re-expressed in terms of the stream function, since we see that in irrotational flow, where our function f vanishes, the stream function is harmonic. We then have

$$\psi = Uy\big(1 - \frac{a^2}{r^2}\big) = U\sin\theta\big(r - \frac{a^2}{r}\big). \tag{3.43}$$

Setting $\psi = h(r)\ \sin\theta$ in $\nabla^2\psi + k^2\psi = 0$ we we obtain the ODE for the Bessel functions of order 1. A solution regular in $r < a$ is therefore $h = CJ_1(kr)$. Ths

$$\psi = C\sin\theta J_1(kr). \tag{3.44}$$

Also

$$\omega = Ck^2\sin\theta J_1(kr). \tag{3.45}$$

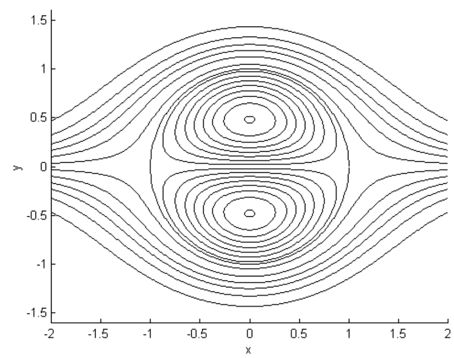

Figure 3.4: A propagating vortex dipole.

We have two constants to determine, and we will do this by requiring that both ω and u_θ be continuous on $r = a$. The condition on ω requires that $J_1(ka) = 0$. We thus choose ka to be the smallest zero if J_1, $ka \approx 3.83$.

The constant C is determined by the requirement that u_θ be continuous on $r = a$. Now $u_\theta = -\sin\theta\ \psi_y - \cos\theta\ \psi_x = -\psi_r$, and

$$\frac{d}{dr}J_1(kr) = -k^{-1}\frac{d^2}{dz^2}J_0(z)\Big|_{z=kr} = k^{-1}\Big(\frac{1}{z}\frac{dJ_0}{dz}+J_0\Big)_{z=kr} = k^{-1}\Big(-\frac{1}{z}J_1+J_0\Big)_{z=kr}. \tag{3.46}$$

Thus

$$\frac{d}{dr}J_1(kr)\Big|_{r=a} = k^{-1}J_0(ka). \tag{3.47}$$

The condition that ψ_r be continuous on $r = a$ thus becomes

$$C = 2k^{-1}\frac{U}{J_0(ak)}. \tag{3.48}$$

Thus

$$\omega = -\nabla^2\psi = \frac{2kU}{J_0(ak)}\sin\theta J_1(kr). \tag{3.49}$$

Since $J_0(3.83) \approx -.403$ we see that the constant multiplier in this last equation has a sign of opposite to that of U. Let us see if this makes sense. If U were negative, then the vorticity in the upper half of the disc would be positive. A positive vorticity implies an eddy rotating counterclockwise. This vorticity induces the vortex in the lower half of the disc to move to the right. Similarly the negative vorticity in the lower half of the disk causes the upper vortex to move to the right. Thus the vortex dipole propagates to the right, and in the frame moving with the dipole U is negative.

3.7.3 Axisymmetric flow

We turn now to a large class of vortical flows which are probably the simplest flows allowing vortex stretching, namely the axisymmetric Euler flows. These are solutions of Euler's equations in cylindrical polar coordinates (z, r, θ), under the assumption that all variables are independent of the polar angle θ. Euler's equations for the velocity $\mathbf{u} = (u_z, u_r, u_\theta)$ in cylindrical polar coordinates are

$$\frac{\partial u_z}{\partial t} + \mathbf{u} \cdot \nabla u_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \tag{3.50}$$

$$\frac{\partial u_r}{\partial t} + \mathbf{u} \cdot \nabla u_r - \frac{u_\theta^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{3.51}$$

$$\frac{\partial u_\theta}{\partial t} + \mathbf{u} \cdot \nabla u_\theta + \frac{u_r u_\theta}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial \theta} = 0, \tag{3.52}$$

where

$$\mathbf{u} \cdot \nabla(\cdot) = \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\Big](\cdot). \tag{3.53}$$

We take the density to be constant, so the solenoidal condition applies in the form

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial r u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0. \tag{3.54}$$

The vorticity vector is given by

$$(\omega_z, \omega_r, \omega_\theta) = \Big[\frac{1}{r}\frac{\partial r u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\Big]. \tag{3.55}$$

The vorticity equation is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \Big[\mathbf{u} \cdot \nabla \omega_z, \mathbf{u} \cdot \nabla \omega_r, \mathbf{u} \cdot \nabla \omega_\theta + \frac{u_\theta \omega_r}{r}\Big] - \Big[\boldsymbol{\omega} \cdot \nabla u_z, \boldsymbol{\omega} \cdot \nabla u_r, \boldsymbol{\omega} \cdot \nabla u_\theta + \frac{u_r \omega_\theta}{r}\Big] = 0. \tag{3.56}$$

In the axisymmetric case we thus have

$$\frac{\partial u_z}{\partial t} + \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}\Big] u_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \tag{3.57}$$

$$\frac{\partial u_r}{\partial t} + \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}\Big] u_r - \frac{u_\theta^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{3.58}$$

$$\frac{\partial u_\theta}{\partial t} + \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}\Big] u_\theta + \frac{u_r u_\theta}{r} = 0, \tag{3.59}$$

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial r u_r}{r} = 0. \tag{3.60}$$

$$(\omega_z, \omega_r, \omega_\theta) = \Big[\frac{1}{r}\frac{\partial r u_\theta}{\partial r}, -\frac{\partial u_\theta}{\partial z}, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\Big] \tag{3.61}$$

If the *swirl* velocity component u_θ vanishes, the system simplifies further:

$$\frac{\partial u_z}{\partial t} + \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}\Big] u_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \tag{3.62}$$

$$\frac{\partial u_r}{\partial t} + \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}\Big] u_r + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{3.63}$$

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial r u_r}{r} = 0. \tag{3.64}$$

$$(\omega_z, \omega_r, \omega_\theta) = \Big[0, 0, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\Big]. \tag{3.65}$$

Note that the only nonzero component of vorticity is ω_θ. The vortex lines are therefore all rings with a common axis, the $z-$axis. The vorticity equation now has the form

$$\frac{\partial \omega_\theta}{\partial t} + u_z \frac{\partial \omega_\theta}{\partial z} + u_r \frac{\partial \omega_\theta}{\partial r} - \frac{u_r \omega_\theta}{r} = 0. \tag{3.66}$$

The last equation may be rewritten

$$\frac{D}{Dt}\frac{\omega_\theta}{r} = 0, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}. \tag{3.67}$$

Thus $\frac{\omega_\theta}{r}$ is a material invariant of the flow. We can easily interpret the meaning of this fact. A vortex ring of radius r has length $2\pi r$, and the vorticity associated with a given ring is a constant ω_θ. But the vorticity of a line is proportional to the line length (recall the increase of vorticity by line stretching). Thus the ratio $\frac{\omega_\theta}{2\pi r}$ must be constant on a given vortex ring. Since vortex rings move with the fluid, $\frac{\omega_\theta}{r}$ is a material invariant.

To compute axisymmetric flow without swirl we can introduce the stream function ψ for the solenoidal velocity in cylindrical polar coordinates:

$$u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}. \tag{3.68}$$

This ψ is often referred to as *the Stokes stream function*. Then

$$\omega_\theta = -\frac{1}{r}L(\psi), \quad L \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}. \tag{3.69}$$

In the *steady* case, the vorticity equation gives

$$\Big[\frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial}{\partial z} - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial}{\partial r}\Big]\frac{1}{r^2}L(\psi) = 0. \tag{3.70}$$

Thus a family of steady solutions can be obtained by solving any equation of the form

$$L(\psi) = r^2 f(\psi), \tag{3.71}$$

where f is an arbitrary function, for the stream function ψ. The situation here is closely analogous to the steady two-dimensional case, see the previous subsection.

Now turning to axisymmetric flow *with* swirl, the instantaneous streamline and vortex lines can now be helices and a much larger class of Euler flows results. The same stream function applies. The swirl velocity satisfies, from (3.59)

$$\frac{Dru_\theta}{Dt} = 0, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_z\frac{\partial}{\partial z} + u_r\frac{\partial}{\partial r}. \tag{3.72}$$

We can understand the meaning of (3.72) using Kelvin's theorem. First note that a ring of fluid particles initially on a given circle C defined by initial values of z, r, will stay on the same circular ring as it evolves. The u_θ component takes the ring into itself, and the $(u_z, u_r, 0)$ sub-field determines the trajectory $C(t)$ of the ring, and thus the ring evolves as a material curve. Since u_θ is constant on the ring, the circulation on $C(t)$ is $2\pi r u_\theta$. By Kelvin's theorem, this circulation is a material invariant, and we obtain (3.72).

In the case of *steady* axisymmetic flow with swirl we see from (3.72) that we may take

$$ru_\theta = g(\psi), \tag{3.73}$$

where the function g is arbitrary. Bernoulli's theorem for steady flow with constant density gives

$$\frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} = H(\psi), \tag{3.74}$$

stating that the Bernoulli function H is constant on streamlines. From the momentum equation in the form $\nabla H - \mathbf{u}\times\boldsymbol{\omega} = 0$ we get, from the z-component e.g.:

$$u_r\omega_\theta - u_\theta\omega_r = \frac{\partial H}{\partial z}. \tag{3.75}$$

Using the expressions for the components of vorticity and expressing everything in terms of the stream function, we get from (3.73) and (3.75)

$$\frac{1}{r^2}\frac{\partial\psi}{\partial z} + \frac{1}{r^2}g\frac{dg}{d\psi}\frac{\partial\psi}{\partial z} = \frac{dH}{d\psi}\frac{\partial\psi}{\partial z}. \tag{3.76}$$

Eliminating the common factor $\frac{\partial\psi}{\partial z}$ and rearranging,

$$L(\psi) = r^2 f(\psi) - g\frac{dg}{d\psi}, \quad f(\psi) = \frac{dH}{d\psi}. \tag{3.77}$$

Thus two arbitrary functions, f, g are involved and any solution of (3.77) determines a steady solution in axisymmetric flow with swirl.

Problem set 3

1. Consider a fluid of constant density in two dimensions with gravity, and suppose that the vorticity $v_x - u_y$ is everywhere constant and equal to ω. Show

that the velocity field has the form $(u, v) = (\phi_x + \chi_y, \phi_y - \chi_x)$ where ϕ is harmonic and χ is any function of x, y (independent of t), satisfying $\nabla^2\chi = -\omega$. Show further that

$$\nabla(\phi_t + \frac{1}{2}q^2 + \omega\psi + p/\rho + gz) = 0$$

where ψ is the stream function for $\mathbf{u}$, i.e. $\mathbf{u} = (\psi_y, -\psi_x)$, and $q^2 = u^2 + v^2$.

2. Show that, for an incompressible fluid, but one where the density can vary independently of pressure (e.g. salty seawater), the vorticity equation is

$$\frac{D\omega}{Dt} = \omega \cdot \nabla\mathbf{u} + \rho^{-2}\nabla\rho \times \nabla p.$$

Interpret the last term on the right physically. (e.g. what happens if lines of constant p are $y =$ constant and lines of constant ρ are $x - y =$constant?). Try to understand how the term acts as a source of vorticity, i.e. causes vorticity to be created in the flow.

3. For steady two-dimensional flow of a fluid of constant density, we have

$$\rho\mathbf{u} \cdot \nabla\mathbf{u} + \nabla p = 0, \nabla \cdot \mathbf{u} = 0.$$

Show that, if $\mathbf{u} = (\psi_y, -\psi_x)$, these equations imply

$$\nabla\psi \times \nabla(\nabla^2\psi) = 0.$$

Thus, show that a solution is obtained by giving a function $H(\psi)$ and then solving $\nabla^2\psi = H'(\psi)$. Show also that the pressure is given by $\frac{p}{\rho} = H(\psi) - \frac{1}{2}(\nabla\psi)^2$+constant.

4. Prove *Ertel's theorem* for a fluid of constant density: If f is a scalar material invariant, i.e. $Df/Dt = 0$, then $\omega \cdot \nabla f$ is also a material invariant, where $\omega = \nabla \times \mathbf{u}$ is the vorticity field.

5. A steady *Beltrami flow* is a velocity field $\mathbf{u}(\mathbf{x})$ for which the vorticity is always parallel to the velocity, i.e. $\nabla \times \mathbf{u} = f(\mathbf{x})\mathbf{u}$ for some scalar function f. Show that if a steady Beltrami field is also the steady velocity field of an inviscid fluid of constant density, the necessarily f is constant on streamlines. What is the corresponding pressure? Show that $\mathbf{u} = (Bsiny + C\cos z, C\sin z + A\cos x, A\sin x + B\cos y)$ is such a Beltrami field with $f = -1$. (This last flow an example of a velocity field yielding chaotic particle paths. This is typical of 3D Beltrami flows with constant f, according to a theorem of V. Arnold.)

6. Another formula exhibiting a vector field $\mathbf{u} = (u, v, w)$ whose curl is $\boldsymbol{\omega} = (\xi, \eta, \zeta)$, where $\nabla \cdot \boldsymbol{\omega} = 0$, is given by

$$u = z\int_0^1 t\eta(tx, ty, tz)dt - y\int_0^1 t\zeta(tx, ty, tz)dt,$$

$$v = x\int_0^1 t\zeta(tx, ty, tz)dt - z\int_0^1 t\xi(tx, ty, tz)dt,$$

$$w = y\int_0^1 t\xi(tx, ty, tz)dt - x\int_0^1 t\eta(tx, ty, tz)dt.$$

Verify this result. (Note that **u** will not in general be divergence-free, e.g. check $\xi = \zeta = 0, \eta = x$. A derivation of this formula, using differential forms, may be found in Flanders' book on the subject.)

7. In this problem the object is to find a 2D propagating vortex dipole structure analogous to that studied in subsection 3.6.2. In the present case, the structure will move clockwise on the circle of radius R with angular velocity Ω. Consider a rotating coordinate system and a circular structure of radius a, stationary and with center at $(0, R)$. Relative to the rotating system the velocity tends to $\Omega(-y, x) = \Omega(-y', x) + \Omega R(-1, 0), y' = y - R$. It turns out that (assuming constant density), the momentum equation relative to the rotating frame can be reduced to that in the non-rotating frame in that the Coriolis force can be absorbed into the gradient of a modified pressure, see a later chapter. Thus we again take $\nabla^2\psi + k^2\psi = 0, r' < a$. Here $r' = \sqrt{(y')^2 + x^2}$. A new term proportional to $J_0(kr)$ must now be included. We require that u_θ aAand ω must be continuous on $r' = a$. Show that, relative to the rotating frame,

$$\psi = \begin{cases} -\frac{2R\Omega}{kJ_0(ak)}\sin\theta J_1(kr') + \frac{2\Omega}{k^2 J_0(ka)}J_0(kr'), & \text{if } r' < a, \\ -\frac{\Omega}{2}r'^2 - \Omega R(r' - a^2/r') + \Omega a^2 \ln r' + C, & \text{if } r' \geq a. \end{cases} \tag{3.78}$$

Chapter 4

Potential flow

Potential or irrotational flow theory is a cornerstone of fluid dynamics, for two reasons. Historically, its importance grew from the developments made possible by the theory of harmonic functions, and the many fluids problems thus made accessible within the theory. But a second, more important point is that potential flow is actually realized in nature, or at least approximated, in many situations of practical importance. Water waves provide an example. Here fluid initially at rest is set in motion by the passage of a wave. Kelvin's theorem insures that the resulting flow will be irrotational whenever the viscous stresses are negligible. We shall see in a later chapter that viscous stresses cannot in general be neglected near rigid boundaries. But often potential flow theory applies away from boundaries, as in effects on distant points of the rapid movements of a body through a fluid.

An example of potential flow in a barotropic fluid is provided by the theory of sound. There the potential is not harmonic, but the irrotational property is acquired by the smallness of the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ in the momentum equation. The latter thus reduces to

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\rho} \nabla p \approx 0. \tag{4.1}$$

Since sound produces very small changes of density, here we may take ρ to be will approximated by the constant ambient density. Thus $\mathbf{u} = \nabla \phi$ with $\frac{\partial \phi}{\partial t} = -p/\rho$.

4.1 Harmonic flows

In a potential flow we have

$$\mathbf{u} = \nabla \phi. \tag{4.2}$$

We also have the Bernoulli relation (for body force $\mathbf{f} = -\rho \nabla \Phi$)

$$\phi_t + \frac{1}{2}(\nabla \phi)^2 + \int \frac{dp}{\rho} + \Phi = 0. \tag{4.3}$$

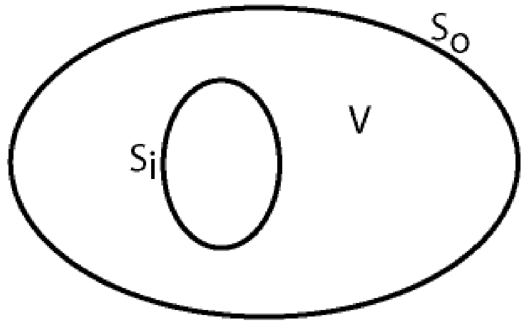

Figure 4.1: A domain V, bounded by surfaces $S_{i,o}$ where $\frac{\partial \phi}{\partial n}$ is prescribed.

Finally, we have conservation of mass

$$\rho_t + \nabla \cdot (\rho \nabla \phi) = 0. \tag{4.4}$$

The most extensive use of potential flow theory is to the case of constant density, where $\nabla \cdot \mathbf{u} = \nabla^2 \phi = 0$. These *harmonic flows* can thus make use of the highly developed mathematical theory of harmonic functions. in the problems we study here we shall usually consider explicit examples where existence is not an issue. On the other hand the question of uniqueness of harmonic flows is an important issue we discuss now. A typical problem is shown in figure 4.1.

A harmonic function ϕ has prescribed normal derivatives on inner and outer boundaries S_i, S_o of an annular region V. The difference $\mathbf{u}_d = \nabla \phi_d$ of two solutions of this problem will have zero normal derivatives on these boundaries. That the difference must in fact be zero throughout V can be established by noting that

$$\nabla \cdot (\phi_d \nabla \phi_d) = (\nabla \phi_d)^2 + \phi_d \nabla^2 \phi_d = (\nabla \phi_d)^2. \tag{4.5}$$

The left-hand side of (4.5) integrates to zero over V to zero by Gauss' theorem and the homogeneous boundary conditions of $\frac{\partial \phi_d}{\partial n}$. Thus $\int_V (\nabla \phi_d)^2 dV = 0$, implying $\mathbf{u}_d = 0$.

Implicit in this proof is the assumption that ϕ_d is a single-value function. A function ϕ is single-valued in V if and only if $\oint_C d\phi = 0$ on any closed contour C contained in V. In three dimensions this is insured by the fact that any such contour may be shrunk to a point in V. In two dimensions, the same conclusion applies to *simply-connected* domains. In non-simply connect domains uniqueness of harmonic flows in 2DS is not assured. Note for a harmonic flow

$$\oint_C d\phi = \oint_C \mathbf{u} \cdot d\mathbf{x} = \Gamma_C, \tag{4.6}$$

so that a potential which is not single valued is associated with a non-zero circulation on some contour. Since there is no vorticity within the domain of harmonicity, we must look outside of this domain to find the vorticity giving rise to the circulation.

Example 4.1: The point vortex of problem 1.2 is an example of a flow harmonic in a non-simply connected domain which excludes the origin. If $\mathbf{u} = \frac{1}{2\pi}(-y/r^2, x/r^2)$ then the potential is $\frac{\theta}{2\pi} + constant$ and the circulation on an simply closed contour oriented counter-clockwise is 1. This defines the *point vortex of unit circulation*. Here the vorticity is concentrated at the origin, outside the domain of harmonicity.

Example 4.2 Steady two-dimensional flow harmonic flow with velocity $(U, 0)$ at infinity, past a circular cylinder of radius a centered at the origin, is not unique. The flow of example 2.4 plus an arbitrary multiple of the point vortex flowof example 4.1 will again yield a flow with the same velocity at infinity, and still tangent to the boundary $r = a$:

$$\phi = Ux(1 + a^2/r^2) + \frac{\Gamma}{2\pi}\theta. \tag{4.7}$$

4.1.1 Two dimensions: complex variables

In two dimensions harmonic flows can be studied with the powerful apparatus of complex variable theory. We define the *complex potential* as an analytic function of the complex variable $z = x + iy$:

$$w(z) = \phi(x, y) + i\psi(x, y). \tag{4.8}$$

We will suppress t in our formulas in the case when the flow is unsteady. If we identify ϕ with the potential of a harmonic flow, and ψ with the stream function of the flow, then by our definitions of these quantities

$$(u, v) = (\phi_x, \phi_y) = (\psi_y, -\psi_x), \tag{4.9}$$

yielding the Cauchy-Riemann equations $\phi_x = \psi_y, \phi_y = -\psi_x$. The derivative of w gives the velocity components in the form

$$\frac{dw}{dz} = w'(z) = u(x, y) - iv(x, y). \tag{4.10}$$

Notice that the Cauchy-Riemann equations imply that $\nabla\phi \cdot \nabla\psi = 0$ at every point where the partials are defined, implying that the streamlines are there orthogonal to the lines of constant potential ϕ.

Example 4.3: The uniform flow at an angle α to the horizontal, with velocity $Q(\cos\alpha, \sin\alpha)$ is given by the complex potential $w = Qze^{-i\alpha}$.

Example 4.4: In complex notation the harmonic flow of example 4.2 may be written

$$w = U(z + a^2/z) - \frac{i\Gamma}{2\pi}\log z \tag{4.11}$$

where e.g. we take the principle branch of the logarithm function.

As a result of the identification of the complex potential with an analytic function of a complex variable, the conformal map becomes a valuable tool in

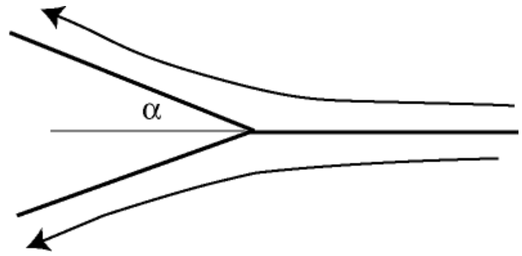

Figure 4.2: Flow onto a wedge of half-angle α.

the construction of potential flows. For this application we may start with the physical of z-plane, where the complex potential $w(z)$ is desired. A conformal map $z \to Z$ transforms boundaries and boundary conditions and leads to a problem which can be solved to obtain a complex potential $W(Z)$. Under the map values of ψ are preserved, so that streamlines map onto streamlines.

Example 4.5: The flow onto a wedge-shaped body (see figure 4.2). Consider in the Z plane the complex potential of a uniform flow,$-UZ, U > 0$. The region above upper surface of the wedge to the left, and the and the positive x-axis to the right, is mapped onto the upper half-plane $Y >$) by the function $Z = z^{\frac{\pi}{\pi-\alpha}}$. Thus $w(z) = -Uz^{\frac{\pi}{\pi-\alpha}}$.

Example 4.6: The map $z(Z) = Z + \frac{b^2}{Z}$ maps the circle of radius $a > b$ in the Z-plane onto the ellipse of semi-major axis $\frac{a^2+b^2}{a}$ and semi-minor (y)-axis $\frac{a^2-b^2}{a}$ in the z-plane. And the exterior is mapped onto the exterior. Uniform flow with velocity $(U, 0)$at infinity, past the circular cylinder $|Z| = a$, has complex potential $W(Z) = U(Z + a^2/Z)$. Inverting the map and requiring that $Z \approx z$ for large $|z|$ gives $Z = \frac{1}{2}(z + \sqrt{z^2 - 4b^2})$. Then $w(z) = W(Z(z))$ is the complex potential for uniform flow past the ellipse. Notice how the map satisfies $\frac{dz}{dZ} \to 1$ as $z \to \infty$ This insures that that infinity maps by the identity and so the uniform flow imposed on the circular cylinder is also imposed on the ellipse.

4.1.2 The circle theorem

We now state a result which gives the mathematical realization of the physical act of "placing a rigid body in an ideal fluid flow", at least in the two-dimensional case.

Theorem 3 *Let a harmonic flow have complex potential $f(z)$, analytic in the domain $|z| \leq a$. If a circular cylinder of radius a is place at the origin, then the new complex potential is $w(z) = f(z) + \overline{f\left(\frac{a^2}{\bar{z}}\right)}$.*

To show this we need to establish that the analytical properties of the new flow match those of the old, in particular that the analytic properties and the singularities in the flow are unchanged. Then we need to verify that the surface of the circle is a streamline. Taking the latter issue first, note that on the circle

$\frac{a^2}{\bar{z}} = z$, so that there we have $w = f(z) + \overline{f(z)}$, implying $\psi = 0$ and so the circle is a streamline. Next, we note that the added term is an analytic function of z if it is not singular at z. (If $f(z)$ is analytic at z, so is $\overline{f(\bar{z})}$. As for the location of singularities of w, since f is analytic in $|z| \leq a$ it follows that $f\left(\frac{a^2}{z}\right)$ is analytic in $|z| \geq a$, and the same is true of $\overline{f\left(\frac{a^2}{\bar{z}}\right)}$. Thus the only singularities of $w(z)$ in $|z| > a$ are those of $f(z)$.

Example 4.7: If a cylinder of radius a is placed in a uniform flow, then $f = Uz$ and $w = Uz + U\overline{(a^2/\bar{z})} = U(z + a^2/z)$ as we already know. If a cylinder is placed in the flow of a point source at $b > a$ on the x-axis, then $f(z) = \frac{Q}{2\pi}\ln(z-b)$ and

$$w(z) = \frac{Q}{2\pi}(\ln(z-b) + \overline{\ln\left(\frac{a^2}{\bar{z}} - b\right)} = \frac{Q}{2\pi}(\ln(z-b) + \ln(z - a^2/b) - \ln z) + C, \quad (4.12)$$

where C is a constant. From this form it may be verified that the imaginary part of w is constant when $z = ae^{i\theta}$. Note that the *image system* of the source, with singularities within the circle, consists of a source of strength Q at the image point a^2/b, and a source of strength $-Q$ at the origin.

Example 4.8: A point vortex at position z_k of circulation Γ_k has the complex potential $w_k(z) = -i\frac{\Gamma_k}{2\pi}\ln(z - z_k)$. A collection of N such vortices will have the potential $w(z) = \sum_{k=1}^{N} w_k(z)$. Since vorticity is a material scalar in two-dimensional ideal flow, and the delta-function concentration may be regarded as the limit of a small circular patch of constant vorticity, we expect that each vortex must move with he harmonic flow created at the vortex by the other $N-1$ vortices. Thus the positions $z_k(t)$ of the vortices under this law of motion is governed by the system of N equations,

$$\overline{\frac{dz_j}{dt}} = \frac{-i}{2\pi}\sum_{k=1, k\neq j}^{N} \frac{\Gamma_k}{z - z_k}. \quad (4.13)$$

Note the conjugation on the left coming from the identity $w' = u - iv$.

4.1.3 The theorem of Blasius

An important calculation in fluid dynamics is the force exerted by the fluid on a rigid body. In two dimensions and in a steady harmonic flow this calculation can be carried out elegantly using the complex potential.

Theorem 4 *Let a steady uniform flow past a fixed two-dimensional body with bounding contour C be a harmonic flow with velocity potential $w(z)$. Then, if no external body forces are present, the force (X, Y) exerted by the fluid on the body is given by*

$$X - iY = \frac{i\rho}{2}\oint_C \left(\frac{dw}{dz}\right)^2 dz. \quad (4.14)$$

Here the integral is taken round the contour in the counter-clockwise sense. This formula, due to Blasius, reduces the force calculation to a complex contour integral. Since the flow is harmonic, the path of integration may be distorted to any simple closed contour encircling he body, enabling the method of residues to be applied. The exact technique will depend upon whether are not the are singularities in the flow exterior to the body.

To prove the result, first recall that $dX - idY = p(-dy - idx) = -ipd\bar{z}$. Also, Bernoulli's theorem for steady ideal flow applies, so that

$$p = -\frac{\rho}{2}\Big|\frac{dw}{dz}\Big|^2 + C, \tag{4.15}$$

where clearly the constant C will play no role. Thus

$$X - iY = \frac{i\rho}{2}\oint_C \frac{dw}{dz}\overline{\frac{dw}{dz}}d\bar{z}. \tag{4.16}$$

However, the contour C is a streamline, so that $d\psi = 0$ there, and so on C we have $\overline{\frac{dw}{dz}}d\bar{z} = d\bar{w} = dw = \frac{dw}{dz}dz$. using this in (4.16) we obtain (4.14).

Example 4.9: We have found in problem 2.1 that the force on a circular cylinder in a uniform flow is zero. To verify this using Blasius' theorem, we set $w = U\Big(z + \frac{a^2}{z}\Big)$ so that $U^2\Big(1 - \frac{a^2}{z^2}\Big)2$ is to be integrated around C. Since there is no term proprotional to z^{-1} in the Laurent expansion about the origin, the residue is zero and we get no contribution to the force integral.

Example 4.10: Consider a source of strength Q placed at $(b, 0)$ and introduce a circular cylinder of radius $a < b$ into the flow. From example 4.6 we have

$$\frac{dw}{dz} = \frac{1}{z-b} + \frac{1}{z - a^2/b} - \frac{1}{z}. \tag{4.17}$$

Squaring, we get

$$\frac{1}{(z-b)^2} + \frac{1}{(z-a^2/b)^2} + \frac{1}{z^2} + \frac{2}{(z-b)(z-a^2/b)} - \frac{2}{z(z-a^2/b)} - \frac{2}{z(z-b)}. \tag{4.18}$$

The first three terms to not contibute to the integral around the circle $|z| = a$. For the last three, the partial fraction decomposition is

$$\frac{A}{z-b} + \frac{B}{z-a^2/b} + \frac{C}{z}, \tag{4.19}$$

where we compute $A = \frac{2a^2}{(b^2-a^2)b}, B = \frac{2b^3}{a^2(a^2-b^2)}, C = \frac{2(a^2+b^2)}{a^2 b}$. The contributions come from the poles within the circle and we have

$$X - iY = \frac{i\rho}{2}\frac{Q^2}{4\pi^2}2\pi i(B + C) = \frac{Q^2\rho}{2\pi}\frac{a^2}{b(b^2-a^2)}. \tag{4.20}$$

The cylinder is therefore feels a force of attraction toward the source.

This introduction to the use of complex variables in the analysis of two-dimenisonal harmonic flows will provide the groundwork for a discussion of lift and airfoil design, to be taken up in chapter 5.

4.2 Flows in three dimensions

We live in three dimensions, not two, and the "flow past body" problem in two dimensions introduces a domain which is not simply connected, with important consequences. The relation between two and three-dimensional flows is particularly significant in the generation of lift, as we shall see in chapter 5. In the present section we treat topics in three dimensions which are direct extensions of the two-dimensional results just given. They pertain to bodies, such as a sphere, which move in an irrotational, harmonic flow.

4.2.1 The simple source

The source of strength Q in three dimensions satisfies

$$\text{div } \mathbf{u} = Q\delta(\mathbf{x}), \quad \mathbf{u} = \nabla\phi. \tag{4.21}$$

Here $\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$ is the three-dimensional delta function. It has the following properties: (i) It vanishes if $\mathbf{x} \neq 0$. (ii) Any integral of $\delta(\mathbf{x})$ over an open region containing the origin yields unity. It is best to think of all relations involving delta functions and other distributions as limits of relations using smooth functions.

In our case, integrating $\nabla^2\phi = Q\delta(\mathbf{x})$ over a sphere of radius $R_0 > 0$ we get

$$\int_{R=R_0} \frac{\partial\phi}{\partial n}\, dS = Q. \tag{4.22}$$

Since $\nabla^2\phi = 0, \mathbf{x} \neq 0$, and since the delta function must be regarded as an isotropic distribution, having no exceptional direction, we make the guess (using now $\nabla^2\phi = R^{-1}d^2(R\phi)/dR^2$) that $\phi = C/R, R^2 = x^2 + y^2 + z^2$ for some constant C. Then (4.22) shows that $C = -\frac{Q}{4\pi}$. Thus the simple source in three dimensions, of strength Q, has the potential

$$\phi = -\frac{Q}{4\pi}\frac{1}{R}. \tag{4.23}$$

Note that Q is equal to the volume of fluid per unit time crossing any deformation of a spherical surface, assuming the deformed surface surounds the origin. [1]

[1] We indicate how to justify this calculation using a limit operation. Define the three-dimensional delta function by $\lim_{\epsilon\to 0}\delta_\epsilon(R)$ where $\delta_\epsilon = \frac{3}{2\pi\epsilon^3}\frac{1}{1+(R/\epsilon)^3}$. Solving $\nabla^2\phi_\epsilon = \delta_\epsilon = R^{-2}\frac{d}{dR}\left(R^2\frac{d\phi_\epsilon}{dR}\right)$, under the condition that ϕ_ϵ vanish at infinity, we obtain $\phi_\epsilon = -\frac{1}{4\pi R} + \int_R^\infty R^{-2}\left[tan^{-1}(R^3\epsilon^{-3}) - \pi/2\right]dR$. For any positive R the integral tends to zero as $\epsilon \to 0$.

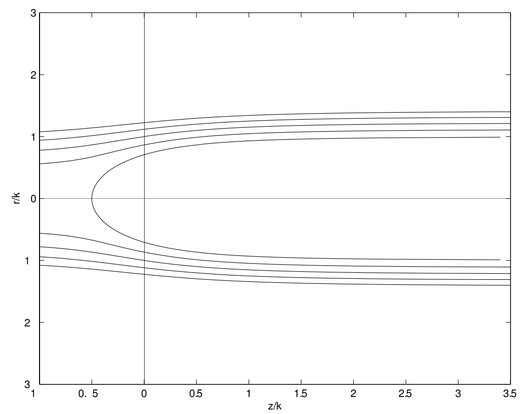

Figure 4.3: The Rankine fairing. All lengths are in units of k.

4.2.2 The Rankine fairing

We consider now a simple source of strength Q placed at the origin in a uniform flow $W\mathbf{i}_z$. The combined potential is then

$$\phi = Uz - \frac{Q}{4\pi}\frac{1}{R}. \tag{4.24}$$

The flow is clearly symmetric about the z-axis. In cylindrical polar coordinates $(z, r, \theta), r^2 = x^2 + y^2$ we introduce again the Stokes stream function ψ:

$$u_z = \phi_z = \frac{1}{r}\frac{\partial\psi}{\partial r},\ u_r = \phi_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}. \tag{4.25}$$

Thus for (4.24) we have

$$\frac{1}{r}\frac{\partial\psi}{\partial r} = U + \frac{Q}{4\pi}\frac{z}{R^3}. \tag{4.26}$$

Integrating,

$$\psi = Ur^2/2 - \frac{Q}{4\pi}\Big(\frac{z}{R} + 1\Big). \tag{4.27}$$

In (4.27) we have chosen the constant of integration to make $\psi = 0$ on the negative z-axis.

We show the stream surface $\psi = 0$, as well as several stream surfaces $\psi > 0$, in figure 4.3. This gives a good example of a uniform flow over a semi-infinite body. An interesting question is whether or not such a body would experience a force. We will find below that D'Alembert's paradox applies to *finite* bodies in three dimensions, that the drag force is zero, but it is not obvious that the result applies to bodies which are not finite.

We will use this question to illustrate the use of conservation of momentum to calculate force on a distant contour. In figure 4.4 the large sphere S of radius R_0 is centered at the origin and intersects the fairing on the at a circle bounding

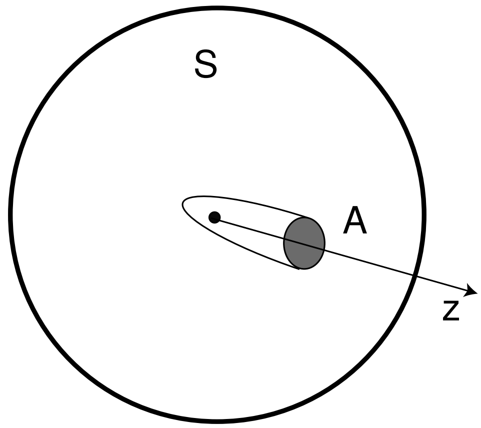

Figure 4.4: Geometry of the momentum integral for computation of the force on the Rankine fairing.

the disc A. Let S' be the spherical surface S minus hat part within the boundary of A. We are considering steady harmonic flow and so the momentum equation may be written

$$\frac{\partial}{\partial x_j}[\rho u_i\, u_j + p\,] = 0. \tag{4.28}$$

Let V' be the region bounded by S' and the piece of fairing enclosed. Integrating (4.28) over V' and using the divergence theorem., the contribution from the surface of the fairing is the integral $-\mathbf{n}p$ over this surface, where $\mathbf{n}$ is the outer normal *of the fairing*. Thus this contribution is the force $\mathbf{F}$ experienced by the enclosed piece of fairing, a force clearly directed along the z axis and therefore equal to the drag, $\mathbf{F} = D\mathbf{i}_z$. The remainder of the integral, taking only the z-component, takes the form of an integral over S minus the contribution from A. Thus conservation of momentum gives

$$D + \rho \int_S u_z \mathbf{u}\cdot\mathbf{R}/R + \frac{1}{2}\left[U^2 - |\mathbf{u}|^2\right]\frac{z}{R}dS - \rho I_A = 0. \tag{4.29}$$

We have here using the Bernoulli formula for the flow, $p + \frac{1}{2}|\mathbf{u}|^2 = \frac{1}{2}U^2$, the pressure at infinity being taken to be zero. Treating first the integral over S, we have

$$\mathbf{u} = U\mathbf{i}_z + \frac{Q}{4\pi}\frac{\mathbf{R}}{R^3}, \quad |\mathbf{u}|^2 = U^2 + \frac{UQ}{2\pi}\frac{z}{R^3} + \frac{Q^2}{16\pi^2}\frac{1}{R^4}. \tag{4.30}$$

Thus the integral in question becomes

$$\int_S \left(U + \frac{Q}{4\pi}\frac{z}{R^3}\right)\left(\frac{Uz}{R} + \frac{Q}{4\pi}\frac{1}{R^2}\right) - \frac{1}{4\pi}\left(UQ\frac{z^2}{R^4} + \frac{1}{8\pi}\frac{Q^2 z}{R^5}\right)dS. \tag{4.31}$$

We see that this last integral gives $UQ + \frac{1}{2}UQ - \frac{1}{2}UQ = UQ$. For the contribution I_A, we take the limit $R_0 \to \infty$ to obtain $I_A = U^2\pi r_\infty^2$, where r_∞ is the

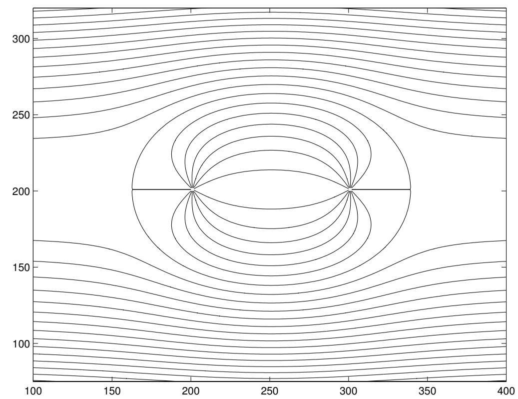

Figure 4.5: Flow around an airship.

asymptotic radius of the airing as $z \to \infty$. In this limit $D' \to D$, the total drag of the fairing. Thus the momentum integral method gives

$$D + \rho(UQ - U^2 \pi r_\infty^2) = 0. \tag{4.32}$$

But from (4.27) we see that the stream surface $\psi = 0$ is given by

$$z = \frac{r^2 - \frac{1}{2}k^2}{\sqrt{k^2 - r^2}}, \quad k^2 = \frac{Q}{\pi U}. \tag{4.33}$$

Thus $r_\infty = k$, and (4.32) becomes

$$D + \rho(UQ - UQ) = D = 0, \tag{4.34}$$

so the drag of the fairing is zero.

Example 4.11: The flow considered now typifies the early attempts to model the pressure distribution of an airship. The model consists of a source of strength Q at position $z = 0$ on the z-axis, and a equalizing sink (source of strength $-Q$) at the pint $z = 1$ on the z-axis. Since the source strengths cancel, a finite body is so defined when the singularities are place in the uniform flow $U\mathbf{i}_z$. It can be shown (see problem 4.7 below), that stream surfaces for the flow are given by constant values of

$$\Psi = \frac{U}{2} R^2 \sin^2\theta - \frac{Q}{4\pi}\Big(\cos\theta + \frac{1 - R\cos\theta}{\sqrt{R^2 - 2R\cos\theta + 1}}\Big), \tag{4.35}$$

where R, θ are spherical polars at the origin, with axial symmetry. We show the stream surfaces in figure 4.4.

4.2.3 The Butler sphere theorem.

The circle theorem for two-dimensional harmonic flows has a direct analog in three dimensions.

Theorem 5 *Consider an axisymmetric harmonic flow in spherical polars* (R, θ, φ), $u_\varphi = 0$, *with Stokes stream function* $\Psi(R, \theta)$ *vanishing at the origin:*

$$u_R = \frac{1}{R^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}, u_\theta = \frac{-1}{R\sin\theta}\frac{\partial \Psi}{\partial R}. \tag{4.36}$$

If a rigid sphere of radius a is introduced into the flow at the origin, and if the singularities of Ψ *exceed a in distance from the origin, then the stream function of the resulting flow is*

$$\Psi_s = \Psi(R, \theta) - \frac{R}{a}\Psi(a^2/R, \theta). \tag{4.37}$$

It is clear that Ψ_s vanishes when $R = a$, so the surface of the sphere is a stream surface. Also the added term introduces no new singularities outside the sphere. Thus the theorrm is proved if we can verify that $\frac{R}{a}\Psi(a^2/R, \theta)$ represents a harmonic flow. In spherical polars with axial symmetry the only component of vorticity is

$$\omega_\varphi = \frac{1}{R}\Big[\frac{\partial (R u_\theta)}{\partial R} - \frac{\partial u_R}{\partial \theta}\Big]. \tag{4.38}$$

Thus the condition on Ψ for an irrotational flow is

$$R^2 \frac{\partial^2 \Psi}{\partial R^2} + \sin\theta \frac{\partial}{\partial \theta}\Big(\frac{1}{\sin\theta}\frac{\partial \Psi}{\partial \theta}\Big) \equiv L_R \Psi = 0. \tag{4.39}$$

If $\frac{R}{a}\Psi(a^2/R, \theta)$ is inserted into (4.39) we can show that the equation is satisfied provided it is satisfied by $\Psi(R, \theta)$, see problem 4.8. Finally, since $\Psi(R, \theta)$ vanishes at the origin at least as R, $R\Psi(a^2/R, \theta$ is bounded at infinity and velocity component must decay as $O(R^{-2})$, so the uniform flow there is undisturbed.

Example 4.12: A sphere in a uniform flow $U\mathbf{i}_z$ has Stokes stream function

$$\Psi(R, \theta) = \frac{U}{2} R^2 \sin^2\theta \Big[1 - \frac{a^3}{R^3}\Big]. \tag{4.40}$$

This translates into the following potential:

$$\phi = Uz\Big(1 + \frac{1}{2}\frac{a^3}{R^3}\Big). \tag{4.41}$$

Example 4.13: Consider a source of strength Q place on the z axis at $z = b$ and place a rigid sphere of radius $a < b$ at the origin. The streamfunction for this source which vanishes at the origin is

$$\Psi(R, \theta) = -\frac{Q}{4\pi}(\cos\theta_1 + 1), \tag{4.42}$$

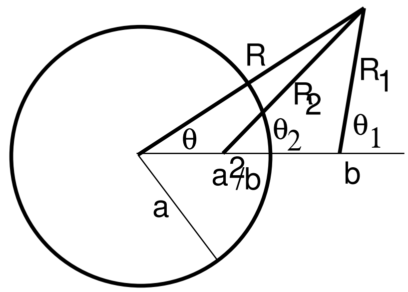

Figure 4.6: A a sphere of radius a in the presence of a source at $z = b > a$.

where θ_1 is defined in figure 4.6.

Now from the law of cosines and figure 4.6 we have

$$R\cos\theta - b = R_1\cos\theta_1, \quad R_1^2 = b^2 - 2bR\cos\theta + R^2. \tag{4.43}$$

Thus

$$\cos\theta_1 = \frac{R\cos\theta - b}{\sqrt{b^2 - 2bR\cos\theta + R^2}}. \tag{4.44}$$

Thus the stream function including the sphere, Ψ_s, is given by

$$\Psi_s = -\frac{Q}{4\pi}\Big[\frac{R\cos\theta - b}{\sqrt{b^2 - 2bR\cos\theta + R^2}} + 1\Big] + \frac{Q}{4\pi}\frac{R}{a}\Big[\frac{\frac{a^2}{R}\cos\theta - b}{\sqrt{b^2 - 2b\frac{a^2}{R}\cos\theta + \frac{a^4}{R^2}}} + 1\Big]. \tag{4.45}$$

Now, again using the law of cosines, $\sqrt{b^2 - 2b\frac{a^2}{R}\cos\theta + \frac{a^4}{R^2}} = bR_2/R$. Also we may use $R^2 = R_2^2 + 2\frac{a^2}{b}R\cos\theta - \frac{a^4}{b^2}$. Then Ψ_s may be brought into the form

$$\Psi_s = -\frac{Q}{4\pi}\Big[\frac{R\cos\theta - b}{R_1} + 1\Big] - \frac{a}{b}\frac{Q}{4\pi}\Big[\frac{R\cos\theta - \frac{a^2}{b}}{R_2}\Big] + \frac{Q}{4\pi}\Big[\frac{R - R_2}{a}\Big]. \tag{4.46}$$

The first term on the right is the source of strength Q at $z = b$. The second term is another source, of strength $\frac{a}{b}Q$, at the image point $z = a^2/b$. The last term can be understood as a line distribution of sinks of density $\frac{Q}{4\pi a}$, extending from the origin to the image point a^2/b. Indeed, if a point P on this line segment is associated with an angle θ_P, the the contribution from such a line of sinks would be

$$\frac{Q}{4\pi a}\int_0^{\frac{a^2}{b}} \cos\theta_P dz. \tag{4.47}$$

But $dR = -\cos\theta_P dz$, so the integral becomes

$$-\frac{Q}{4\pi a}\int_R^{R_2} dR = \frac{Q}{4\pi a}(R - R_2). \tag{4.48}$$

4.3 Apparent mass and the dynamics of a body in a fluid

Although harmonic flow is an idealization never realized exactly in actual fluids (except in some cases of super fluid dynamics), it is a good approximation in many fluid problems, particularly when rapid changes occur. A good example is the abrupt movement of a solid body through a fluid, for example a swimming stroke of the hand. We know from experience that a abrupt movement of the hand through water gives rise to a force opposing the movement. It is easy to see why this must be, within the theory of harmonic flows. An abrupt movement of the hand through still water causes the fluid to move relative to a observer fixed with the still fluid at infinity. This observer would therefore compute at the instant the hand is moving a finite kinetic energy of the fluid, whereas before the movement began the kinetic energy was zero. To produce this kinetic energy work must have been done, and so a force with a finite component opposite to the direction of motion must have occurred. We are here dealing only with the fluid, but if the body has mass the clearly a force is also needed to accelerate that mass. Thus both the body mass and the fluid movement contribute to the force experienced.

In a harmonic flow we shall show that, in the absence of external body forces, the force on a rigid body is proportional to its acceleration, and further the force contributed by the fluid can be expressed as an addition, *apparent* mass of the body. In other words the augmented force due to the presence of the surrounding fluid and the energy it acquires during motion of a body, can be explained as an inertial force associated with additional mass and the work done against that force. The term *virtual mass* is also used to denote this apparent mass. For a sphere, which has an isotropic geometry with no preferred direction, the apparent mass is just a scalar to be added to the physical mass. In general, however, the apparent mass associated with the momentum of a body in two or three dimensions will depend on the direction of the velocity vector. It thus must be a second order tensor, represented by the *apparent mass matrix*.

4.3.1 The kinetic energy of a moving body

Consider an ideal fluid at rest and introduce a moving rigid body, in two or three dimensions. An observer at rest relative to the fluid at infinity will se a disturbance of the flow which vanishes at infinity. It would be natural to compute compute the momentum of this motion by calculating the integral $\int \rho \mathbf{u} dV$ of the region exterior to the body. The problem is that such harmonic flows have an expansion at infinity of the form

$$\phi \sim a \ln r - \mathbf{A} \cdot \mathbf{r} r^{-2} + O(r^{-2}) \tag{4.49}$$

in two dimensions and

$$\phi \sim \frac{a}{R} - \mathbf{A} \cdot \mathbf{R} R^{-3} + O(R^{-3}) \tag{4.50}$$

in three dimensions. Thus

$$\rho \int \nabla \phi dV = \int_S \phi \mathbf{n} dS, \tag{4.51}$$

where S comprises both a surface in a neighborhood of infinity as well as the body surface, is not absolutely convergent as the distant surfaces recedes. We point out that $a = 0$ in two dimensions if the area of the body is fixed and there is no circulation about the body. In three dimensions a vanishes if the body has fixed volume, see problem 4.12.

But even if $a = 0$ and $\phi = O(R^{-1})$ the value of the integral is only conditionally convergent will depend on how one defines the distant surface. So the value attributed to the fluid momentum is ambiguous by this calculation.

An unambiguous result is however possible, if we instead focus on the kinetic energy and from it determine the incremental momentum created by a change in velocity. Let us fix the orientation of the body and consider its movement through space, without rotation. This *translation* is completely determined by a velocity vector $\mathbf{U}(t)$. The, from the discussion of section 2.6 we know that a harmonic flow will satisfy the instantaneous boundary condition

$$\frac{\partial \phi}{\partial n} = \mathbf{U}(t) \cdot \mathbf{n} \tag{4.52}$$

on the surface of the body. Now $\nabla^2 \phi = 0$ is a linear equation, and so we see that there must exist a vthe Φ_i as encoding the effect of the shape of the body from all possible harmonic flows associated with translation of the body.

We may now compute the kinetic energy E of the fluid exterior to the body using

$$\mathbf{u} = U_i \nabla \Phi_i. \tag{4.53}$$

Thus

$$E(t) = \frac{1}{2} M_{ij} U_i U_j, \quad M_{ij} = \rho \int \nabla \Phi_i \cdot \nabla \Phi_j dV. \tag{4.54}$$

the integral being over the fluid domain. Clearly the matrix M_{ij} is symmetric, and thus

$$dE = M_{ij} U_j dU_i. \tag{4.55}$$

On the other hand the change of kinetic energy, dE, must equal, in the absence of external body forces, the work done by the force $\mathbf{F}$ which the body exerts on the fluid, $dE = \mathbf{F} \cdot \mathbf{U} dt$. But according to Newton's second law, the incremental momentum $d\mathbf{P}$ is given by $d\mathbf{P} = \mathbf{F} dt$. Consequently $dE = \mathbf{U} \cdot d\mathbf{P}$. From (4.54) we thus have

$$dE = \rho M_{ij} dU_j U_i = dP_i U_i. \tag{4.56}$$

Since this holds for arbitrary $\mathbf{U}$ we must have $dP_i = M_{ij} dU_j$. Integrating and using the fact that M_{ij} is independent of time and $\mathbf{P} = 0$ when $\mathbf{U} = 0$ we obtain

$$P_i = M_{ij} U_j. \tag{4.57}$$

Thus we have reduced the problem of computing momentum, and then the inertial force, to calculating M_{ij}. Since M_{ij} arises here as an effective mass term associated with movement of the body, it is called the *apparent mass matrix.*

But the calculation of M_{ij} is not ambiguous since the integral for the kinetic energy converges absolutely, and we can deduce M_{ij} once the energy is written in the form (4.54). We write

$$E = \frac{\rho}{2}\int_V |\mathbf{u}|^2 dV = \frac{\rho}{2}\int_V (\mathbf{u}-\mathbf{U})\cdot(\mathbf{u}+\mathbf{U})dV + \frac{\rho}{2}\int_V |\mathbf{U}|^2 dV. \tag{4.58}$$

The reason for this splitting is to exhibit $\mathbf{u}-\mathbf{U}$, whose normal component will vanish on the body by (4.52). Now $\mathbf{u}+\mathbf{U} = \nabla(\phi+\mathbf{U}\cdot\mathbf{x}$ and $\mathbf{u}-\mathbf{U}$ is solenoidal, so $\mathbf{u}-\mathbf{U})\cdot(\mathbf{u}+\mathbf{U}\nabla\cdot\big[(\phi+\mathbf{U}\cdot\mathbf{x})(\mathbf{u}-\mathbf{U})\big]$. Thus, remembering that $|\mathbf{U}|^2$ is a constant, the application of the divergence theorem and use of (4.52) on the inner boundary aallows us to reduce (4.58) to

$$E = \frac{\rho}{2}\int_{S_o} (\phi+\mathbf{U}\cdot\mathbf{x})(\mathbf{u}-\mathbf{U})\cdot\mathbf{n}dS + |\mathbf{U}|^2(\mathcal{V}-\mathcal{V}_b), \tag{4.59}$$

where S_0 is the outer boundary, $\mathcal{V}$is the volume contained by S_o, and $\mathcal{V}_0$ is the volume of the body.

To compute the integral in (4.59) we need only the leading term of ϕ. Referring to (4.49),(4.50), we note that $a = 0$ for a finite rigid body (or even for a flexible body of constant area/volume), see problem 4.11. Using

$$\phi = -\frac{\mathbf{A}\cdot\mathbf{x}}{|\mathbf{x}|^N}, \quad \mathbf{u} = \frac{-\mathbf{A}}{|\mathbf{x}|^N} + \frac{N\mathbf{A}\cdot\mathbf{x}\,\mathbf{x}}{|\mathbf{x}|^{N+2}} \tag{4.60}$$

in (4.59) we have

$$E \sim \frac{\rho}{2}\int_{S_o} \Big[\frac{-\mathbf{A}\cdot\mathbf{x}}{|\mathbf{x}|^N} + \mathbf{U}\cdot\mathbf{x}\Big]\Big[\frac{-\mathbf{A}}{|\mathbf{x}|^N} + \frac{N\mathbf{A}\cdot\mathbf{x}\,\mathbf{x}}{|\mathbf{x}|^{N+2}} - \mathbf{U}\Big]\cdot\mathbf{n}dS. \tag{4.61}$$

We are free to choose S_o to be a sphere of radius R_o. The term quadratic in $\mathbf{A}$ in (4.61) is $O(R_o^{1-2N}$)and so the contribution is of order R_o^{-N} and will vanish in the limit. The term under the integral quadratic in $\mathbf{U}$ yields $-|\mathbf{U}|^2\mathcal{V}$, thus canceling part of the last term in (4.59). Finally two of the cross terms in $\mathbf{U}, \mathbf{A}$ cancel out, the remaining term giving rthe contribution $2\pi\rho(N-1)\mathbf{A}\cdot\mathbf{U}$. Thus

$$E = \frac{\rho}{2}\big[2\pi\rho(N-1)\mathbf{A}\cdot\mathbf{U} - \mathcal{V}_b|\mathbf{U}|^2\big]. \tag{4.62}$$

Since $\phi = \Phi_i U_i$, we may write $A_i(t) = \rho^{-1}m_{ij}U_j$ where m_{ij} is dependent on body shape but not time. Then

$$E = \frac{1}{2}\big[2\pi(N-1)m_{ij} - \mathcal{V}_b\rho\delta_{ij}\big]U_iU_j. \tag{4.63}$$

Comparing (4.63) and (4.54) we obtain an expression for the apparent mass matrix:

$$M_{ij} = 2\pi(N-1)m_{ij} - \mathcal{V}_b\rho\delta_{ij}, \quad N = 2, 3. \tag{4.64}$$

We thus can obtain the apparent mass of a body by a knowledge of the expansion of ϕ in a neighborhood of infinity.

Given that we have computed a finite fluid momentum we are in a position to state

Theorem 6 *(D'Alembert's paradox) In a steady flow of a perfect fluid in three dimensions, and in steady flow in two dimensions for a body with zero circulation, the force experienced by the body is zero.*

Clearly if the flow is steady $d\mathbf{P}/dt = \mathbf{F} = 0$, and we are done. Of course the proof hinges on the existence of a finite fluid momentum associated with a single-value potential function.

Example 4.14: To find the apparent mass matrix of a elliptic cylinder in two dimensions, we may use example 4.6. In the Z-plane the complex potential for uniform flow $-Q(\cos\theta, \sin\theta)$ past the cylinder of radius $a > b$ is $W(Z) = -Qe^{-i\theta}Z - Qe^{i\theta}s^a/Z$. Since $Z = \frac{1}{2}(z + \sqrt{z^2 - 4b^2})$ we may expand at infinity to get

$$w(z) \sim -Qe^{-i\theta}z - Q\Big[\frac{a^2e^{i\theta} - b^2e^{-i\theta}}{z}\Big] + \dots, \tag{4.65}$$

so that

$$\mathbf{A} = \big[U(a^2 - b^2), V(a^2 + b^2)\big],\ \ (U, V) = Q(\cos\theta, \sin\theta). \tag{4.66}$$

Now the ellipse intersects the positive x-axis at its semi-major axis $\alpha = \frac{a^2+b^2}{a}$, and the positive y-axis at its semi-minor axis $\beta = \frac{a^2-b^2}{a}$. From (4.66) we obtain the apparent mass matrix

$$\mathbf{M} = 2\pi\rho\begin{pmatrix} a^2 - b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix} - \pi\frac{a^4 - b^4}{a^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \pi\rho\begin{pmatrix} \beta^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}. \tag{4.67}$$

In particular for a circular cylinder the apparent mass is just the mass of of the fluid displaced by the body.

An alternative expression for the apparent mass matrix in terms of an integral over the surface of the body rather than a distant surface is readily obtained in terms of the potential Φ_i. We have

$$E = \frac{\rho}{2}\int_V \nabla\Phi_i \cdot \nabla\Phi_j U_i U_j dV = \frac{\rho}{2}\int_V \nabla \cdot \Phi_j \nabla\Phi_i] dV U_i U_j. \tag{4.68}$$

Applying the divergence theorem to the integral, surfaces S_o, s_B, and observing that $\Phi_i\nabla\Phi_j = O(|\mathbf{x}|^{1-2N})$, we see that the receding surface integral will give zero contribution. Recalling that $\frac{\partial\phi}{\partial n} = \mathbf{U}\cdot\mathbf{n}$ on the body surface, we see that $\frac{\partial\Phi_i}{\partial n} = n_i$ *where the normal is directed out of the body surface.* In applying the divergence theorem the normal at the body is into the body, with the result that (4.54) applies with

$$M_{ij} = -\rho\int_{S_b} \Phi_j n_i dS,\ \ \mathbf{n}\text{ directed out of the body.} \tag{4.69}$$

It follows from (4.57) that the fluid momentum is given by

$$\mathbf{P} = -\rho \int_{S_b} \phi \mathbf{n} dS. \tag{4.70}$$

We can verify the fact that (4.70) gives the fluid momentum by taking its time derivative, using the result of problem 1.6:

$$\frac{d}{dt} \int_{S_b} \phi \mathbf{n} dS = \int_{S_b} \frac{\partial \phi}{\partial t} \mathbf{n} dS + \int_{S_b} (\mathbf{u} \cdot \mathbf{n}) \nabla \phi dS. \tag{4.71}$$

Using the Bernoulli theorem for harmonic flow we have

$$\frac{d}{dt} \int_{S_b} \phi \mathbf{n} dS = \int_{S_b} \Big[-\frac{p}{\rho} - \frac{1}{2} |\mathbf{u}|^2 \Big] \mathbf{n} - (\mathbf{u} \cdot \mathbf{n}) \mathbf{u} dS. \tag{4.72}$$

Converting the terms on the right involving $\mathbf{u}$ to a volume integral, we observe that the latter converges absolutely at infinity, as so we have, for the integration over the domain exterior to S_b,

$$\int_V \big[\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{2} \nabla |\mathbf{u}|^2 \big] dV = - \int_V \mathbf{u} \times (\nabla \times \mathbf{u}) dV = 0. \tag{4.73}$$

Therefore

$$-\frac{d}{dt} \rho \int_{S_b} \phi \mathbf{n} dS = \int_{S_b} p \mathbf{n} dS = \mathbf{F}, \tag{4.74}$$

where $\mathbf{F}$ is the force applied by the body to the fluid.

Finally we note again that the inertial force required to accelerate a body in a perfect fluid will contain a contribution from the actual mass of the body, M_b. This mass appears as an additional term $M_b \delta_{ij}$ in the expression (4.64) for the apparent mass matrix. The total momentum of the body including its apparent mass is thus $P_i = M_{ij} U_j + M_b U_i$ and Newton's second law becomes

$$\frac{dP_i}{dt} = F_i, \tag{4.75}$$

where $\mathbf{F}$ is the force applied to the body, to accelerate it and the surrounding fluid.

4.3.2 Moment

We have so far restricted the motion of the body to translation, i.e. with no rotation relative to the fluid at infinity. In general a moment is experienced by a body in translational motion, so that in fact a free body will rotate and thereby give the apparent mass matrix a dependence upon time. The theory may be easily extended to include a time dependent apparent mass, due either to rotation and/or deformation of he body, see section 4.4. But even in steady translational motion of a body, a non-zero moment can result, see problem 4.14. (There is no D'Alembert paradox for moment.)

For example, in analogy with (4.69), the *apparent angular momentum* of the fluid exterior to a body is defined by

$$\mathbf{P}_A = -\rho \int_{S_b} \phi(\mathbf{x} \times \mathbf{n}) dS, \tag{4.76}$$

the normal being out of the body. It may be shown in a manner similar to that used for linear momentum that

$$\frac{d\mathbf{P}_A}{dt} = \mathbf{T}, \tag{4.77}$$

where $\mathbf{T}$ is the torque applied to the fluid by the body.

4.4 Deformable bodies and their locomotion

It might be thought that, in an ideal, or more suggestively, a "slippery" fluid, it would be impossible for a body to locomote, i.e. to "swim" by using some kind of mechanism involving changes of shape. The fact is, however, that inertial forces alone can allow a certain kind of locomotion. The key point is that the flow remains irrotational everywhere, and this will have the effect of disallowing the possibility of the body producing an average force on the fluid which can then accelerate the body. Rather, it is possible to locomote in the sense of getting from point A to point B, put without any finite average acceleration. If the body is assumed to deform periodically over some cycle of configurations, then the kind of locomotion we envision is of a finite, periodic translation (and possible rotation) of the body, repeated with each cycle of deformation.

We first note that the Newtonian relationships that we derived above for a rigid body carry over to an arbitrary deformable body, which for simplicity we take to have a fixed area/volume. This follows immediately from our verification of $\frac{d\mathbf{P}}{dt} = \mathbf{F}$ from (4.70), since we made no assumption about the velocity of the body surface.

Now the idea behind inertial swimming is to deform the body in a periodic cycle which causes a net translation. To simplify the problem we consider only a simple traslation of a suitable symmetric body along a line, e.g a body symmetric about the z-axis, translating with velocity $U(t)$along this axis. In general we cannot expect the velocity to remain of one sign, but over one cycle there will be a positive translation, say to the right. Let $U_m(t)$ be the velocity of the center of mass of the body, and let $U_v(t)$ be the velocity of the center of volume of the body. Also let P_D be the momentum of deformation of the body *relative to its center of volume*. If the total mass of the body is m, then $U_m(t)m$ is the momentum of the body mass. Consider now the momentum of the fluid. If the apparent mass of the body (now a scalar $M(t)$) is multiplied by $U_v(t)$, we get the fluid momentum associated with the instantaneous motion of the shape of the body at time t. Finally, we have the momentum associated with the motion of the boundary of the body relative to the center of volume. If the potential of this harmonic flow of deformation is ϕ_D, then the deformation

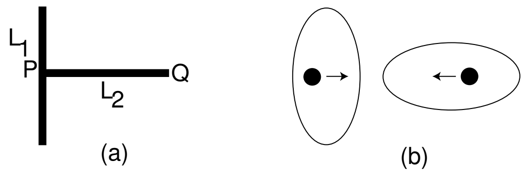

Figure 4.7: Swimming in an ideal fluid.

momentum is $P_D(t) = -\rho \int_{S_b} \phi_D \mathbf{n} \cdot \mathbf{i} dS$. The total momentum if body and fluid is thus $mU_m(t) + M(t)U_v(t) + P_D$. If initially the fluid and body is at rest, then this momentum, which is conserved, must vanish, and it is for this reason that locomotion is possible.

Consider first a body of uniform density. so the center of mass and of volume coincide. The $U_m = U_v = U$ and

$$U(t) = \frac{P_D(t)}{m + M(t)}. \tag{4.78}$$

There is no reason for the right-hand side o (4.78) to have non-zero time average, and when it does not, we call this *locomotion by squirming*. To see squirming in action it is best to treat an simple case, see example 4.15 below.

Alternatively, we can imagine that the center of mass changes relative to the center of volume without and deformation. Then deformation occurs giving a new shape, then the center of mass again changes relative to the center of volume holding the boy fixed in the new shape. If the two shapes lead to different apparent masses, locomotion occurs by *recoil swimming*, see example 4.16.

Example 4.15: We show in figure 4.7(a) a squirming body of a simplified kind. The body consists of a thin vertical strip of length $L_1(t)$, and a horizontal part of length $L_2(t)$. The length will change as a function of time, think of L_2 as being extruded from the material of L_1. We neglect the width w of the strips except when computing mass and volume. The latter are constant, implying $L_1 + L_2 = L$ is constant. The density of the material is taken as ρ_b, so the total mass is $M_b = \rho_b w L$ and the total volume is wL.

A cycle begins with $L_1 = L$, when L_2 begins to grow to the right. If $X(t)$ denotes the position of the point P, then $(\rho_b w L_1 + \pi\rho L_1^2/4)\frac{dX}{dt}$ is the momentum of the fluid and vertical segment, where we have used the formula for apparent mass of a flat plate in 2D. The velocity of the extruded strip varies linearly from $\frac{dX}{dt}$ at P to $\frac{d(X+L_2)}{dt}$ at Q, so the momentum of the horizontal part is $\rho_b w L_2 d(X + \frac{1}{2}L_2)/dt$, where we neglect the apparent mass of the extruded strip. The first half of the cycle stops when $L_1 = 0$. Assuming the start is from fluid and body at rest, the sum of these momenta remains zero throughout the half-cycle:

$$(\rho_b w L + \pi\rho L_1^2/4)\frac{dX}{dt} + \frac{\rho_b w L_2}{2}\frac{dL_2}{dt} = 0. \tag{4.79}$$

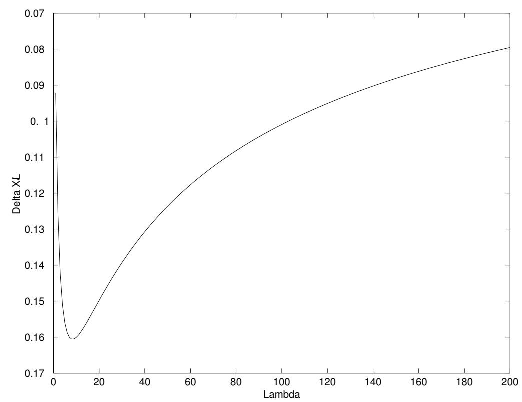

Figure 4.8: $\Delta X/L$ versus λ for the model squirmer of figure 4.7(a).

If we et$L_2 = Lt/T, L_1 = L(1-t/T)$ where T is the half-period of the cycle, then we may obtain the change ΔX of X over the half-cycle by quadrature:

$$\Delta(X) = \frac{1}{\lambda}\ln(1+\lambda) - \frac{1}{\sqrt{\lambda}}\tan^{-1}(\sqrt{\lambda}), \quad \lambda = \frac{\pi\rho L}{4\rho_b w}. \tag{4.80}$$

We show this relation in figure 4.8 So we see that at the end of the half-cycle the point P has moved a distance $-\Delta X$ to the left. At this point, we imagine another half-cycle in which L_1 is created at the expense of L_2, but *at the point* Q. Observe that at the start of the second half-cycle Q is located a distance $L + \Delta X$ from the initial position of P. It can be seen from considerations of symmetry that the point Q will move to the left a distance $-\Delta X$ in time T over the second half-cycle. The the cycle is complete, $L_2 = L$, and the midpoint can be relabeled P. Thus the net advance to the right of the point P in time $2T$ has been $L + 2\Delta X$, which from figure 4.8 always exceeds about $.68L$.

Example 4.16: Recoil swimming can be illustrated by the 2D model of Figure (4.7)b. Let P denote the center of an elliptical surface of major,minor semi-axes α, β. Within this body is a mass M on a bar enabling it to be driven to the right or left. The weight of the shell and mechanism is m. Let the position of the center be $X(t)$ and the position of the mass be $x(t)$. At the beginning of the half-cycle the mass lies a distance $\beta/2$ to the right of P and the ellipse has its major axis vertical. The mass the moves to the left a distance β. Since momentunm is conserved, we have

$$(m + \rho\pi\alpha^2)\frac{dX}{dt} + M(\frac{d(X+x)}{dt} = 0. \tag{4.81}$$

Thus over a half-cycle $(m + \rho\pi\alpha^2)\Delta X + M(\Delta X + \Delta x) = 0$ or, since $\Delta x = \beta$,

$$\Delta X_1 = -\frac{M\beta}{m + M + \rho\pi\alpha^2}. \tag{4.82}$$

at this point the surface of the body deforms in a symmetric way, the points $(0, \pm\alpha/2$ moving down to $(0, \pm\beta/2$ and the points $(\pm\beta/2, 0$ moving our to $(\pm\alpha/2, 0)$, so that the major and minor axes get interchanged. There is no movement of P during this process. No the mass is moved back, a distance β to the left. We see that in this second half-cycle the displacement is

$$\Delta X_2 = \frac{M\beta}{m + M + \rho\pi\beta^2}. \tag{4.83}$$

The displacement over one cycle is then

$$\Delta X = \Delta X_1 + \Delta X_2 = \frac{M\beta}{m + M + \rho\pi\beta^2} - \frac{M\beta}{m + M + \rho\pi\alpha^2}, \tag{4.84}$$

which is positive since $\beta < \alpha$.

Problem set 4

1. (a) Show that the complex potential $w = Ue^{i\alpha}z$ determines a uniform flow making an angle α with respect to the x-axis and having speed U.

(b) Describe the flow field whose complex potential is given by

$$w = Uze^{i\alpha} + \frac{Ua^2e^{-i\alpha}}{z}.$$

2. Recall the system (4.13) governing the motion of point vortices in two dimensions. (a) Using these equations, show that two vortices of equal strengths rotate on a circle with center at the midpoint of the line joining them, and find the speed of their motion.

(b) Show that two vortices of strengths γ and $-\gamma$ move together on straight parallel lines perpendicular to the line joining them. Again find the speed of their motion.

3. Using the method of Blasius, show that the moment of a body in 2D potential flow, about the axis perpendicular to the plane (positive counter-clockwise), is given by

$$M = -\frac{1}{2}\rho Re[\int_C z(dw/dz)^2 dz],$$

where Re denotes the real part and C is any simple closed curve about the body. Using this, verify by the residue method that the moment on a circular cylinder with a point vortex of circulation Γ at its center, in uniform flow, experiences zero moment.

4. Compute, using the Blasius formula, the force exerted by a point vortex at the point $c = be^{i\theta}, b > a$ upon a circular cylinder at the origin of radius a. The complex potential of a point vortex at c is $\frac{-\Gamma i}{2\pi}\ln(z-c)$. (Use the circle theorem and residues). Verify that the cylinder is pushed away from the vortex.

5. Prove Kelvin's minimum energy theorem: In a simply-connected domain V let $\mathbf{u} = \nabla\phi, \nabla^2\phi = 0$, with $\partial\phi/\partial n = f$ on the boundary S of V. (This $\mathbf{u}$ is unique in a simply-connected domain). If $\mathbf{v}$ is any differentiable vector field satisfying $\nabla \cdot \mathbf{v} = 0$ in V and $\mathbf{v} \cdot \mathbf{n} = f$ on S, then

$$\int_V |\mathbf{v}|^2 dV \geq \int_V |\mathbf{u}|^2 dV.$$

(Hint: Let $\mathbf{v} = \mathbf{u} + \mathbf{w}$, and apply the divergence theorem to the cross term.)

6. Establish (4.33) and work though he details of the proof of zero drag of the Rankine fairing using the momentum integral method, as outlined in section 4.2.2.

7. In spherical polar coordinates (r, θ, φ) a Stokes stream function Ψ may be defined by $u_R = \frac{1}{R^2 \sin\theta}\frac{\partial\Psi}{\partial\theta}, u_\theta = \frac{-1}{R\sin\theta}\frac{\partial\Psi}{\partial R}$.) Show that in spherical polar coordinates, the stream function Ψ for a source of strength Q, placed at the origin, normalized so that $\Psi = 0$ on $\theta = 0$, is given by $\Psi = \frac{Q}{4\pi}(1-\cos\theta)$. Verify that the stream function in spherical polars for the airship model consisting of equal source and sink of strength Q, the source at the origin and the sink at $R = 1, \theta = 0$, in a uniform stream with stream function $\frac{1}{2}UR^2(\sin\theta)^2$, is given by (4.35). (Suggestion: The sink will involve the angle with respect to $R = 1, \theta = 0$. Use the law of cosines ($c^2 = a^2 + b^2 - 2ab\cos\theta$ for a triangle with θ opposite side c) to express Ψ in terms of R, θ.)

8. In the Butler sphere theorem, we needed the following result: Show that $\Psi_1(R,\theta) \equiv \frac{R}{a}\Psi(\frac{a^2}{R}, \theta)$ is the stream function of an irrotational, axisymmetric flow in spherical polar cordinates, provided that $\Psi(R,\theta)$ is such a flow. (Hint: Show that $L_R\Psi_1(R,\theta) = L_{R_1}\Psi(R_1,\theta)$, where $R_1 = a^2/R$. Here L_R is defined by (4.39).)

9. (Reading, Milne-Thomson sec. 13.52 on "stationary vortex filaments in the presence of a circular cylinder" in 3rd edition.) Consider the following model of flow past a circular cylinder of radius a with two eddies downstream of the body. Consider two point vortices, of opposite strengths, the upper vortex having clockwise circulation $-\Gamma$(i.e. $\Gamma > 0$) located at the point $c = be^{i\theta}$, thus adding a term $(i\Gamma/2\pi)\ln(z-c)$ to the complex potential w, the other being having circulation Γ at the point $\bar{c} = be^{-i\theta}$. Here $b > a > 0$.

Using the circle theorem, write down the complex potential for the entire flow field, and determine by differentiation the complex velocity. Sketch the positions of the vortices and all vortex singularities within the cylinder, indicating their strengths.

10. Continuing problem 9, verify that $x = \pm a, y = 0$ remain stagnation points of the flow. Show that the vortices will remain stationary behind the cylinder (i.e. not move with the flow) provided that

$$U(1 - \frac{a^2}{c^2}) = \frac{i\Gamma}{2\pi}\frac{(c^2 - a^2)(b^2 - a^2) + (c - \bar{c})^2 a^2}{(c - \bar{c})(c^2 - a^2)(b^2 - a^2)}.$$

Show (by dividing both sides of the last equation by their conjugates and simplifying the result) that this relation implies $b - a^2/b = 2b\sin\theta$, that is, the distance between the exterior vortices is equal to the distance between a vortex and its image vortex.

11. Show that the apparent mass matrix for a sphere is $M_0/2\delta_{ij}$ where M_0 is the mass of fluid displaced by the sphere.

12. Show that for a body which may have a time-dependent shape but is of fixed area/volume, the quantity a in (4.49),(4.50) must vanish.

13. Using the alternative definition (4.69), show that M_{ij} is a symmetric matrix.

14. Let the elliptic cylinder of examples 4.14 and 5.13 be place in a steady uniform flow (U, V). Show, using the result of problem 4.3, that the moment experienced by the cylinder is $-\pi\rho UV(\alpha^2 - \beta^2)$, α, β being the major and minor semi-axes of the ellipse.