Chapter 3

Vorticity

We have already encountered the vorticity field in the identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \frac{1}{2} |\mathbf{u}|^2 - \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{3.1}$$

The vorticity field $\boldsymbol{\omega}(\mathbf{x}, t)$ is defined from the velocity field by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}. \tag{3.2}$$

A potential flow is a flow with zero vorticity. The term *irrotational flow* is widely used. According to (3.1) the contribution to the acceleration coming from the gradient of velocity can be split into two components, one having a potential $\frac{1}{2}|\mathbf{u}|^2$, the other given as a cross product orthogonal to both the velocity and the vorticity. The latter component in older works in fluid dynamics has been called the *vortex force*.

We remark that, in analogy with stream lines, we shall refer to the flow lines of the vorticity field, i.e. the integral curves of the system

$$\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}, \tag{3.3}$$

as (instantaneous) *vortex lines*. Similarly, in analogy with a stream tube in three dimensions, we will refer to a bundle of vortex lines a *vortex tube*.

This straightforward definition of the vorticity field gives little insight into its importance, either physically and theoretically. This chapter will be devoted to examining the vorticity field from a variety of viewpoints.

3.1 Local analysis of the velocity field

The first thing to be noted is that vorticity is fundamentally an Eulerian property since it involves spatial derivatives of the Eulerian velocity field. In a sense the analytical structure of the flow is being expanded to include the first derivatives of the velocity field. Suppose we expand the velocity field in a Taylor series about the fixed point $\mathbf{x}$:

$$u_i(\mathbf{x}+\mathbf{y},t) = u_i(\mathbf{x},t) + y_j \frac{\partial u_i}{\partial x_j}(\mathbf{x},t) + O(|\mathbf{y}|^2). \tag{3.4}$$

We can make the division

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big] + \frac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big]. \tag{3.5}$$

The term first term on the right, $\frac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big]$, is often denoted by e_{ij} and is the *rate-of-strain tensor* of the fluid. Here it will play a basic role when viscous stresses are considered (Chapter 5). The second term, $\frac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big]$, can be seen to be, in three dimensions, the matrix

$$\mathbf{\Omega} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{3.6}$$

Example 3.1: In two dimensions, since u, v depend only on x, y, only one component of the vorticity is non-zero, $\omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. This is usually written simply as the scalar ω. Consider the two-dimensional flow $(u, v) = (y, 0)$. In this case

$$\mathbf{e} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{\Omega} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.7}$$

and $\omega = -1$. This is a simple "shear flow" with horizontal particle paths. Both $\mathbf{e}$ and $\mathbf{\Omega}$ are non-vanishing.

Example 3.2: Consider the flow $(u, v) = (-y, x)$. This is a simple solid-body rotation in the anti-clockwise sense. The vorticity is $\omega = 2$, and $\mathbf{e} = 0$.

These examples are a bit atypical because the vorticity is constant, but they emphasize that a close association of the vorticity with fluid rotation, a connection suggested by the skew-symmetric form of $\mathbf{\Omega}$, can be misleading.

Vorticity is a *point* property, but can only be defined by the limit operations implicit in the needed derivatives. So it is impossible to attach a physical meaning to "the vorticity of a particle". We *can* truncate (3.4) and consider the Lagrangian paths of fluid particles near $\mathbf{x}$. Since $\mathbf{e}$ is real symmetric, it may be diagonalized by a rotation to principal axes. Let the eigenvalues along the diagonal be $\lambda_i, i = 1, 2, 3$. We the may assume our coordinate system is such that $\mathbf{e}$ is the diagonal matrix $\mathbf{D}(\mathbf{x})$. Then the Lagrangian coordinates of the perturbed path $\mathbf{y}$ satisfies

$$\mathbf{y}_t = \mathbf{D}(\mathbf{x})\mathbf{y} + \frac{1}{2}\boldsymbol{\omega}(\mathbf{x}) \times \mathbf{y}. \tag{3.8}$$

These equations couple together the rotation associated with the vorticity at $\mathbf{x}$ with the straining field described by the first term. Note that the angular velocity associated with second term is $\frac{1}{2}\boldsymbol{\omega}$. The statement " vorticity at $\mathbf{x}$ equals twice the angular velocity of the fluid at $\mathbf{x}$" is often heard. But this statement in fact makes no sense, since an angular velocity cannot be attributed to a point. *Given the velocity field of a fluid, one can determine the effects of vorticity on the fluid only on a small open set, i.e. a fluid parcel.*

On the other hand it is true that when vorticity is sufficiently large there is sensible rotation observed in the fluid, and it *is* true that when one sees "rotation" in the fluid, then vorticity is present. In a sense this is the key to understanding its role, since it forces a definition of "rotation" in a fluid.

3.2 Circulation

Let C be a simple, smooth, oriented closed contour which is a deformation of a circle, hence the boundary of an oriented surface S. Now Stokes' theorem applied to the velocity field states that

$$\int_C \mathbf{u} \cdot d\mathbf{x} = \int_S \mathbf{n} \cdot (\nabla \times \mathbf{u}) dS, \tag{3.9}$$

where the direction of the normal $\mathbf{n}$ to S is chosen from the orientation of C by the "right-hand rule". We can interpret the right-hand side of (3.9) as the *flux of vorticity through S.* So it must be that the left-hand side is an expression of the effect of vorticity *on the velocity field*. We thus define *the fluid circulation of the velocity field* $\mathbf{u}$ *on the contour* C by

$$\Gamma_C = \oint_C \mathbf{u} \cdot d\mathbf{x}. \tag{3.10}$$

The circulation is going to be our measure of the rotation of the fluid.

The key "point" is that is that circulation is defined *globally*, *not* at a point. We need to consider an open set containing S in order to make this definition.

Example 3.3: Potential flows have the property that circulation vanishes on any closed contour, as long as $\mathbf{u}$ is well-behaved in an open set containing S. This is an obvious property of an irrotational flow.

Example 3.4 In two dimensions, the flow $(u, v) = \frac{1}{2\pi}(-y/r^2, x/r^2)$ is a point vortex. If C is a simple closed curve encircling the origin, then Γ_C is equal to the circulation on a circle centered at the origin, by independence of path since (u, v) is irrotational everywhere except at the origin. The circulation on a circle, taken counter-clockwise, is found to be unity. Indeed in polar form the velocity is given by $u_r = 0, u_\theta = \frac{1}{2\pi r}$. The circulation on the circle of radius r is thus $\frac{2\pi r}{2\pi r} = 1$. This flow is called the *point vortex of unit strength.*

3.3 Kelvin's theorem for a barotropic fluid

In chapters 12-14 we will be taking up the dynamics of general compressible fluids. The intervening discussion will deal with only a restricted class of compressible flows, the *barotropic fluids.* A barotropic fluid is defined by specifying pressure as a given function of the density, $p(\rho)$. This reduces the dependent variables of an ideal fluid to $\mathbf{u}, \rho$ and so the system of momentum and mass equations is closed.

Theorem 1 *(Kelvin's theorem) Let $C(t)$ be a simple close* material *curve in an ideal fluid with body force $-\rho\nabla\Phi$. Then, if either (i) $\rho =$ constant, or (ii) the fluid is barotropic, then the circulation $\Gamma_{C(t)}$ of $\mathbf{u}$ on C is invariant under the flow:*

$$\frac{d}{dt}\Gamma_{C(t)} = 0. \tag{3.11}$$

To prove this consider a parametrization $\mathbf{x}(\alpha, t)$ of $C(t)$, $0 \le \alpha \le A$. Then

$$\frac{d}{dt}\oint_C \mathbf{u}\cdot d\mathbf{x} = \frac{d}{dt}\int_0^A \mathbf{u}\cdot\frac{\partial \mathbf{x}}{\partial\alpha}d\alpha = \int_0^A \Big[\frac{D\mathbf{u}}{Dt}\cdot\frac{\partial \mathbf{x}}{\partial\alpha} + \mathbf{u}\cdot\frac{\partial \mathbf{u}}{\partial\alpha}\Big]d\alpha. \tag{3.12}$$

Making use of the momentum equation $\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$ we have

$$\frac{d\Gamma_C}{dt} = \int_0^A \Big[-(\frac{1}{\rho}\nabla p + \nabla\Phi)\cdot\frac{\partial \mathbf{x}}{\partial\alpha} + \mathbf{u}\cdot\frac{\partial \mathbf{u}}{\partial\alpha}\Big]d\alpha, \tag{3.13}$$

This becomes

$$\frac{d\Gamma_C}{dt} = \oint_C \Big[\frac{-dp}{\rho} + d(\frac{1}{2}|\mathbf{u}|^2 - \Phi)\Big]. \tag{3.14}$$

Now if ρ is a constant, or if the fluid is barotropic, the integrand may be written as perfect differential (in the barotropic case a differential of $-\int \rho^{-1}\frac{dp}{d\rho}d\rho + \frac{1}{2}|\mathbf{u}|^2 - \Phi$). Since all variables are assumed single-valued, the integral vanishes and the theorem is proved.

Kelvin's theorem is a cornerstone of ideal fluid theory since it expresses a global property of vorticity, namely the flux through a surface, as an invariant of the flow. We shall see that it is very useful in understanding the kinematics of vorticity.

3.4 The vorticity equation

In the present section we again assume that either $\rho =$ constant, or else the fluid is barotropic.

In either case it is of interest to consider an equation for vorticity, which can be obtained by taking the curl of

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0. \tag{3.15}$$

Under the conditions stated, this will give

$$\nabla\times\frac{D\mathbf{u}}{Dt} = 0. \tag{3.16}$$

Recalling $\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u} \times \boldsymbol{\omega}$, we use the vector identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B} + \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A}. \tag{3.17}$$

For the case of constant density and no mass addition, both $\nabla \cdot \mathbf{u}$ and $\nabla \cdot \boldsymbol{\omega}$ vanish, with the result

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}. \tag{3.18}$$

For a barotropic fluid, we need to bring in conservation of mass to evaluate $\nabla \cdot \mathbf{u} = -\rho^{-1} D\rho/Dt$. We then get in place of (3.18)

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \frac{\boldsymbol{\omega}}{\rho}\frac{D\rho}{Dt}. \tag{3.19}$$

This can be rewritten as

$$\frac{D(\frac{\boldsymbol{\omega}}{\rho})}{Dt} = \frac{\boldsymbol{\omega}}{\rho} \cdot \nabla \mathbf{u}. \tag{3.20}$$

Now we want to compare (3.18) and (3.20) with (1.23), and observe that $\boldsymbol{\omega}$ in the first case and $\boldsymbol{\omega}/\rho$ is the second is a *material vector field* as we defined it in chapter 1. This is a deep and remarkable property of the vorticity field, which gives it its importance in fluid mechanics. It tells us, for example, that vorticity magnitude can be increased if two nearby fluid particles lying on the same vortex line move apart.

Example 3.5 In two dimensions $\boldsymbol{\omega} \cdot \nabla \mathbf{u} = 0$ and so the vorticity ω satisfies

$$\frac{D\omega}{Dt} = 0, \tag{3.21}$$

i.e. in two dimensions, for the cases studied here, vorticity is a scalar material invariant, whose value is always the same on a given fluid parcel.

In three dimensions the term $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$ is sometimes called the *vortex stretching term*. Its existence make two and three-dimensional vorticity behaviors entirely different.

There is a Lagrangian form of the vorticity equation, due to Gauss. We can obtain it here by recalling that $v_i(\mathbf{a}, t) = J_{ij}(\mathbf{a}, t)V_i(\mathbf{a})$ defines a material vector field. Let us assume that, given the initial velocity and therefore initial vorticity fields, vorticity may be solved for uniquely at some function time t using Euler's equations. Then, any material vector field assuming the assigned initial values for vorticity must be the unique vorticity field $\boldsymbol{\omega}$. However, if the initial vorticity is $\boldsymbol{\omega}_0(\mathbf{x})$, then a material vector field which takes on these initial values is $\mathbf{J}(\mathbf{a}, t) \cdot \boldsymbol{\omega}_0(\mathbf{a})$. By uniqueness, we must have

$$\omega_i(\mathbf{a}, t) = J_{ij}(\mathbf{a}, t)\omega_{0j}. \tag{3.22}$$

in the constant density case. For the barotropic case, given initial density $\rho_0(\mathbf{x})$, the corresponding equation is

$$\rho^{-1}\omega_i(\mathbf{a}, t) = \rho_0^{-1}(\mathbf{a})J_{ij}(\mathbf{a}, t)\omega_{0j}. \tag{3.23}$$

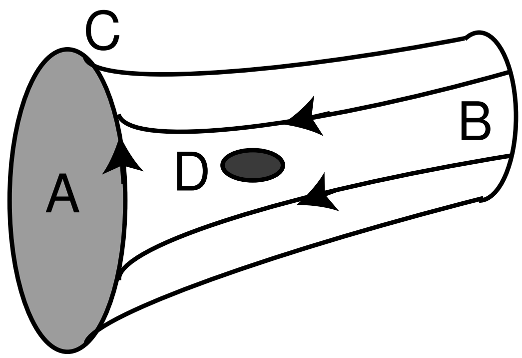

Figure 3.1: A segment of an oriented vortex tube.

This is Cauchy's "solution" of the vorticity equation . Of course nothing has been solved, only represented in terms of the unknown Jacobian. It is however a revealing relation which directly ties the changes in vorticity to the deformation experienced by a fluid parcel.

3.5 Helmholtz' Laws

In dicussing the behavior of vorticity in a fluid flow we will want to consider as our basic element a section of a vortex tube as shown in figure 3.1. Recall that a vortex tube is a bundle of vortex lines, each of the lines being the instantaneous flow lines of of the vorticity field.

In the mid-nineteenth century Helmholtz laid the foundations for the mechanics of vorticity. His conclusions can be summarized by the following three laws:

- Fluid parcels free of vorticity stay free of vorticity.
- Vortex lines are material lines.
- The strength of a vortex tube, to be defined below, is an invariant of the motion.

We have seen that the vorticity field, or the field divided by density in the barotropic case, is a material vector field. The vortex lines are the same in each case if $\boldsymbol{\omega}$ is the same. Hence particles on a particular vortex line at one time, remain on a line at a later time, and so the line is itself material. Thus the tube segment in figure 3.1 is bounded laterally by a surface of vortex lines. The small patch D in the surface thus carries no flux of vorticity. The bounding contour of this patch is a material curve, and by Kelvin's theorem the circulation on the contour is a material invariant. Since this circulation is initially zero by the absence of flux of vorticity through the patch, it will remain zero. Consequently the lateral boundary of a vortex tube remains a boundary of the tube.

In follows from the solenoidal property of vorticity and the divergence theorem that the flux of vorticity through the end surface A, must equal that through the end surface B. This flux is a property of a vortex tube, called the *vortex tube strength*. Note that this is independent of the compressibility or incompressibility of the fluid. The tube strength expresses simply a property of a solenoidal vector field.

To establish the third law of Helmholtz we must show that this strength is a material invariant. But this follows immediately from Kelvin's theorem, since the circulation on the contour C is a material invariant. This circulation, for the orientation of the contour shown in the figure, is equal to the vortex tube strength by Stokes' theorem, and we are done.

The first law is also established using Kelvin's theorem. Suppose that a flow is initially irrotational but at some time a fluid parcetionl is found where vorticity is non-zero. A small closed contour can then be found with non-vanishing circula is non-zero, by Kelvin's theorem. This contradicts the irrotationality of the initial flow.

Using these laws we may see how changes in the shape of a fluid parcel can change the magnitude of vorticity. In figure 3.2 we show a segment of small vortex tube which has changes under the flow from have length $L_1$and section area A_1, to new values A_2, L_2. If the density is constant, volume is conserved, $A_1L_1 = A_2L_2$. If the vorticity magnitudes are ω_1, ω_2, then invariance of the tube strength implies $\omega_1 A_1 = \omega_2 A_2$. Comparing these expressions, $\omega_2/\omega_1 = L_2/L_1$. Consequently, *for an ideal fluid of constant density the vorticity is proportional to vortex line length*. We understand here that by line length we are referring to the distance between to nearby fluid particles on the same vortex line. Thus the growth or decay of vorticity in ideal fluid flow is intimately connected to the stretching properties of the Lagrangian map.[1] Fluid tubulence is observed to contain small domains of very large vorticity, presumably created by this stretching.

For a compressible fluid the volume of te tube need not be invariant, but mass is conserved. Thus we have, introducing the initaial and final densities ρ_1, ρ_2,

$$\rho_1 A_1 L_1 = \rho_1 A_2 L_2, \quad \omega_1 A_1 = \omega_2 A_2. \tag{3.24}$$

It follows that

$$\frac{\omega_2/\rho_2}{\omega_1/\rho_1} = L_2/L_1. \tag{3.25}$$

Thus we see that it is the magnitude of the material field, whether $|\boldsymbol{\omega}|$ or $\rho^{-1}|\boldsymbol{\omega}|$, which is proportional to line length. Notice that in a compressible fluid vorticity may be increased by compressing a tube while holding the length fixed, so as to increase the density.

[1] This make chaotic flow, with positive Liapunov exponents, of great interest in amplifying vorticity.

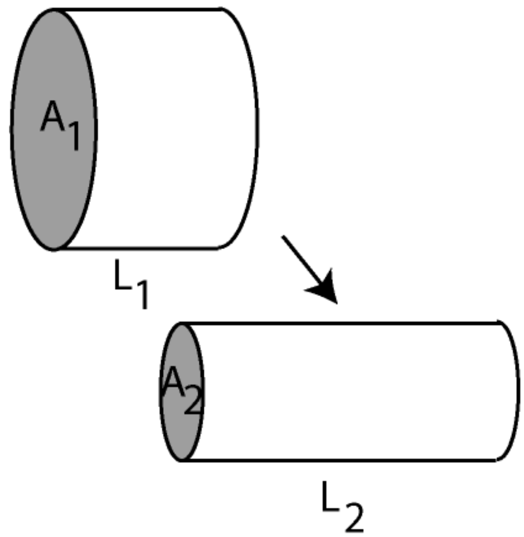

Figure 3.2: Deformation of a vortex tube under a flow.

3.6 The velocity field created by a given vorticity field

Suppose that in R^3 the vorticity field is non-zero in some region and vanishes at infinity. What is the velocity field or fields is created by this vorticity? It is clear that given a vorticity field $\boldsymbol{\omega}$, and a vector field $\mathbf{u}$ such that $\nabla \times \mathbf{u} = \boldsymbol{\omega}$, another vector field with the same property is given by $\mathbf{v} = \mathbf{u} + \nabla\phi$ for some scalar field ϕ, uniqueness is an issue. However, under appropriate conditions a unique construction is possible.

Theorem 2 *Let the given vorticity field be smooth and vanish strongly at infinity, e.g. for some* $R > 0$

$$|\boldsymbol{\omega}| \leq Cr^{-N}, \quad r > R, r = \sqrt{x^2 + y^2 + z^2} \tag{3.26}$$

Then there exists a unique solenoidal vector field $\mathbf{u}$ *such that* $\nabla \times \mathbf{u} = \boldsymbol{\omega}$ *and* $\lim_{r\to\infty} |\mathbf{u}| = 0$. *This vector field is given by*

$$\mathbf{u} = \frac{1}{4\pi} \int_{R^3} \frac{(\mathbf{y} - \mathbf{x}) \times \boldsymbol{\omega}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} dV_{\mathbf{y}}. \tag{3.27}$$

To prove this consider the vector field $\mathbf{v}$ defined by

$$\mathbf{v} = \frac{1}{4\pi} \int_{R^3} \frac{\boldsymbol{\omega}}{|\mathbf{x} - \mathbf{y}|} dV_{\mathbf{y}}. \tag{3.28}$$

This field exists and given (3.26) and can be differentiated if $\boldsymbol{\omega}$ is a smooth function. Let $\mathbf{u} = \nabla \times \mathbf{v}$. Now we have the vector identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}. \tag{3.29}$$

The right-hand side of (3.28) is the unique solution of the vector equation $\nabla^2 \mathbf{v} = \boldsymbol{\omega}$ which vanishes at infinity. Also

$$\begin{aligned} \operatorname{div} \int_{R^3} \boldsymbol{\omega} \cdot \nabla_{\mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{y}|} dV_{\mathbf{y}} &= -\int_{R^3} \boldsymbol{\omega} \cdot \nabla_{\mathbf{y}} \frac{1}{|\mathbf{x} - \mathbf{y}|} dV_{\mathbf{y}} \\ &= -\int_{R^3} \nabla_{\mathbf{y}} \cdot \Big[\frac{\boldsymbol{\omega}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \Big] dV_{\mathbf{y}} = 0 \end{aligned} \tag{3.30}$$

by the divergence theorem and the fact that the integral of $|\boldsymbol{\omega}|$ over $r = R$ be bounded in R in (3.26) holds. Thus $\mathbf{u}$ as defined by (3.27) satisfies $\nabla \times \mathbf{u} = \boldsymbol{\omega}$. as required. Also, this vector field is solenoidal since it is the curl of $\mathbf{v}$, and vanishes as $|\mathbf{x}| \to \infty$. And it is unique. Indeed if $\mathbf{u}'$ is another vector field with the same properties, then $\nabla \times (\mathbf{u} - \mathbf{u}') = 0$ and so $\mathbf{u} - \mathbf{u}' = \nabla\phi$ for some scale field whose gradient vanishes at infinity. But by the solenoidal property of $\mathbf{u}, \mathbf{u}'$ we see that $\nabla^2\phi = 0$, and this implies $\phi = \textit{constant}$, giving the uniqueness of $\mathbf{u}$.

For compressible flows a general velocity field $\mathbf{w}$ with vorticity $\boldsymbol{\omega}$ will have the form $\mathbf{w} = \mathbf{u} + \nabla\phi$ where $\mathbf{u}$ is given by (3.27) and ϕ is an arbitrary scalar field.

The kernel

$$\frac{1}{4\pi} \frac{(\mathbf{y} - \mathbf{x}) \times (\cdot)}{|\mathbf{x} - \mathbf{y}|^3} \tag{3.31}$$

is interesting in the insight it gives into the creation of velocity as a cross product operation. The velocity induced by a small segment of vortex tube is orthogonal to both the direction of the tube and the vector joining the observation point to the vortex tube segment. A similar law relates magnetic field created by an electric cucrrent, where it is know as the *Biot-Savart law*.

3.7 Some examples of vortical flows

We end this chapter with a few examples of ideal fluid flows with non-zero vorticity.

3.7.1 Rankine's combined vortex

This old example is an interesting use of a vortical flow to model a "bath tub vortex", before the depression of the surface of the fluid develops a "hole". It will also give us an example of a flow with a free surface. The fluid is a liquid of constant density ρ with a free surface given by $z = Z(r)$ in cylindrical polar coordinates, see figure 3.3. The pressure above the free surface is the constant p_0. The body force is gravitational, $\mathbf{f} = -g\mathbf{i}_z$. The vorticity is a solid-body rotation in a vertical tube bounded by $r = a, z < Z$. The only nonzero velocity component it the θ-component u_θ.

In $r > a, z < Z$ Euler's equations will be solved by the field of a two-dimensional point vortex (actually a *line vortex*). This will be matched with a rigid rotation for $r < a$ so that velocity is continuous:

$$u_\theta = \begin{cases} \Omega a^2/r, & \text{when } r \geq a, \\ \Omega r, & \text{when } r < a. \end{cases} \tag{3.32}$$

Here Ω is the angular velocity of the core vortex. Now in the exterior region $r > a$ the flow is irrotational and so we have by the Bernoulli theorem for irrotational flows

$$\frac{p_{ext}}{\rho} = \frac{p_0}{\rho} - \frac{1}{2}\Omega^2 a^4 r^{-2} - gz, \tag{3.33}$$

for $z < Z$, where we have taken $Z = 0$ at $r = \infty$. The free surface is thus given for $r > a$ by

$$Z = -\frac{\Omega^2 a^4}{gr^2}. \tag{3.34}$$

Inside the vortex core, the equations reduce to

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{u_\theta^2}{r} = \Omega^2 r, \quad \frac{1}{\rho}\frac{\partial p}{\partial z} = g. \tag{3.35}$$

Thus

$$\frac{1}{2}\Omega^2 r^2 - gz + C \equiv \frac{p_{core}}{\rho}, \quad r < a, z < Z. \tag{3.36}$$

On the cylinder $r = a, z < Z$ we require that the $p_{core} = p_{ext}$, so

$$\frac{1}{2}\Omega^2 a^2 - gz + C = \frac{p_0}{\rho} - gz - \frac{1}{2}\Omega^2 a^2. \tag{3.37}$$

Therefore the constant C is given by

$$C = \frac{p_0}{\rho} - \Omega^2 a^2, \tag{3.38}$$

and

$$\frac{p_{core}}{\rho} = \frac{p_0}{\rho} - \Omega^2 a^2\Big(1 - \frac{r^2}{2a^2}\Big) - gz. \tag{3.39}$$

The free surface is then given by

$$Z = \begin{cases} -\frac{a^4\Omega^2}{2gr^2}, & \text{when } r \geq a, \\ \frac{\Omega^2 a^2}{g}\Big(\frac{r^2}{2a^2} - 1\Big), & \text{when } r < a./\text{cr} \end{cases} \tag{3.40}$$

We have used the adjective "combined" to emphasize that this vortex flow is an example of a solution of the equations of motions which is not smooth, since du_θ/dr is not continuous at $r = a, z < Z$. Since all other components

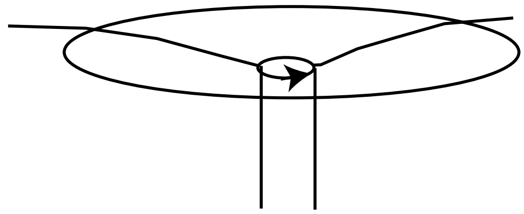

Figure 3.3: Rankine's combined vortex

of velocity are zero and the pressure is the only variable with a z-dependence, the equation are in fact satisfied everywhere. In a real, viscous fluid, if the ideal flow was taken as an initial condition, the irregularity at $r = a$ would be immediately smoothed out by viscous stresses. The ideal fluid solution would nonetheless be a good representation of the flow for some time, until the vortex core is substantially affected by the viscosity.

3.7.2 Steady propagation of a vortex dipole

We consider steady two-dimensional flow of an ideal fluid of constant density, no body force. Since then $\mathbf{u} \cdot \nabla\omega = 0$, introducing the stream function ψ, $(u, v) = (\psi_y, -\psi_x)$, we have

$$\psi_y(\nabla^2\psi)_x - \psi_x(\nabla^2\psi)_y = 0. \tag{3.41}$$

Consequently contours of constant ψ and of constant ω must agree, and so

$$\nabla^2\psi = f(\psi). \tag{3.42}$$

where the function f is arbitrary. We will look for solutions of the simplest kind, by choosing $f = -k^2\psi$, where k is a constant. Using polar coordinates, we look for solutions of the equation $\nabla^2\psi + k^2\psi$ in the disc $r < a$, which can match with the velocity in $r > a$ that is the same as irrotational flow past a circular body of radius a. That potential flow is easily re-expressed in terms of the stream function, since we see that in irrotational flow, where our function f vanishes, the stream function is harmonic. We then have

$$\psi = Uy\big(1 - \frac{a^2}{r^2}\big) = U\sin\theta\big(r - \frac{a^2}{r}\big). \tag{3.43}$$

Setting $\psi = h(r)\ \sin\theta$ in $\nabla^2\psi + k^2\psi = 0$ we we obtain the ODE for the Bessel functions of order 1. A solution regular in $r < a$ is therefore $h = CJ_1(kr)$. Ths

$$\psi = C\sin\theta J_1(kr). \tag{3.44}$$

Also

$$\omega = Ck^2\sin\theta J_1(kr). \tag{3.45}$$

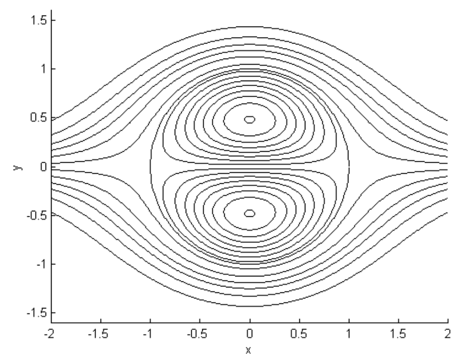

Figure 3.4: A propagating vortex dipole.

We have two constants to determine, and we will do this by requiring that both ω and u_θ be continuous on $r = a$. The condition on ω requires that $J_1(ka) = 0$. We thus choose ka to be the smallest zero if J_1, $ka \approx 3.83$.

The constant C is determined by the requirement that u_θ be continuous on $r = a$. Now $u_\theta = -\sin\theta\,\psi_y - \cos\theta\,\psi_x = -\psi_r$, and

$$\frac{d}{dr}J_1(kr) = -k^{-1}\frac{d^2}{dz^2}J_0(z)\Big|_{z=kr} = k^{-1}\Big(\frac{1}{z}\frac{dJ_0}{dz}+J_0\Big)_{z=kr} = k^{-1}\Big(-\frac{1}{z}J_1+J_0\Big)_{z=kr}. \tag{3.46}$$

Thus

$$\frac{d}{dr}J_1(kr)\Big|_{r=a} = k^{-1}J_0(ka). \tag{3.47}$$

The condition that ψ_r be continuous on $r = a$ thus becomes

$$C = 2k^{-1}\frac{U}{J_0(ak)}. \tag{3.48}$$

Thus

$$\omega = -\nabla^2\psi = \frac{2kU}{J_0(ak)}\sin\theta J_1(kr). \tag{3.49}$$

Since $J_0(3.83) \approx -.403$ we see that the constant multiplier in this last equation has a sign of opposite to that of U. Let us see if this makes sense. If U were negative, then the vorticity in the upper half of the disc would be positive. A positive vorticity implies an eddy rotating counterclockwise. This vorticity induces the vortex in the lower half of the disc to move to the right. Similarly the negative vorticity in the lower half of the disk causes the upper vortex to move to the right. Thus the vortex dipole propagates to the right, and in the frame moving with the dipole U is negative.

3.7.3 Axisymmetric flow

We turn now to a large class of vortical flows which are probably the simplest flows allowing vortex stretching, namely the axisymmetric Euler flows. These are solutions of Euler's equations in cylindrical polar coordinates (z, r, θ), under the assumption that all variables are independent of the polar angle θ. Euler's equations for the velocity $\mathbf{u} = (u_z, u_r, u_\theta)$ in cylindrical polar coordinates are

$$\frac{\partial u_z}{\partial t} + \mathbf{u} \cdot \nabla u_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \tag{3.50}$$

$$\frac{\partial u_r}{\partial t} + \mathbf{u} \cdot \nabla u_r - \frac{u_\theta^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{3.51}$$

$$\frac{\partial u_\theta}{\partial t} + \mathbf{u} \cdot \nabla u_\theta + \frac{u_r u_\theta}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial \theta} = 0, \tag{3.52}$$

where

$$\mathbf{u} \cdot \nabla(\cdot) = \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\Big](\cdot). \tag{3.53}$$

We take the density to be constant, so the solenoidal condition applies in the form

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial r u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0. \tag{3.54}$$

The vorticity vector is given by

$$(\omega_z, \omega_r, \omega_\theta) = \Big[\frac{1}{r}\frac{\partial r u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\Big]. \tag{3.55}$$

The vorticity equation is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \Big[\mathbf{u} \cdot \nabla \omega_z, \mathbf{u} \cdot \nabla \omega_r, \mathbf{u} \cdot \nabla \omega_\theta + \frac{u_\theta \omega_r}{r}\Big] - \Big[\boldsymbol{\omega} \cdot \nabla u_z, \boldsymbol{\omega} \cdot \nabla u_r, \boldsymbol{\omega} \cdot \nabla u_\theta + \frac{u_r \omega_\theta}{r}\Big] = 0. \tag{3.56}$$

In the axisymmetric case we thus have

$$\frac{\partial u_z}{\partial t} + \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}\Big] u_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \tag{3.57}$$

$$\frac{\partial u_r}{\partial t} + \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}\Big] u_r - \frac{u_\theta^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{3.58}$$

$$\frac{\partial u_\theta}{\partial t} + \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}\Big] u_\theta + \frac{u_r u_\theta}{r} = 0, \tag{3.59}$$

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial r u_r}{r} = 0. \tag{3.60}$$

$$(\omega_z, \omega_r, \omega_\theta) = \Big[\frac{1}{r}\frac{\partial r u_\theta}{\partial r}, -\frac{\partial u_\theta}{\partial z}, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\Big] \tag{3.61}$$

If the *swirl* velocity component u_θ vanishes, the system simplifies further:

$$\frac{\partial u_z}{\partial t} + \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}\Big] u_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \tag{3.62}$$

$$\frac{\partial u_r}{\partial t} + \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}\Big] u_r + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{3.63}$$

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial r u_r}{r} = 0. \tag{3.64}$$

$$(\omega_z, \omega_r, \omega_\theta) = \Big[0, 0, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\Big]. \tag{3.65}$$

Note that the only nonzero component of vorticity is ω_θ. The vortex lines are therefore all rings with a common axis, the $z-$axis. The vorticity equation now has the form

$$\frac{\partial \omega_\theta}{\partial t} + u_z \frac{\partial \omega_\theta}{\partial z} + u_r \frac{\partial \omega_\theta}{\partial r} - \frac{u_r \omega_\theta}{r} = 0. \tag{3.66}$$

The last equation may be rewritten

$$\frac{D}{Dt}\frac{\omega_\theta}{r} = 0, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}. \tag{3.67}$$

Thus $\frac{\omega_\theta}{r}$ is a material invariant of the flow. We can easily interpret the meaning of this fact. A vortex ring of radius r has length $2\pi r$, and the vorticity associated with a given ring is a constant ω_θ. But the vorticity of a line is proportional to the line length (recall the increase of vorticity by line stretching). Thus the ratio $\frac{\omega_\theta}{2\pi r}$ must be constant on a given vortex ring. Since vortex rings move with the fluid, $\frac{\omega_\theta}{r}$ is a material invariant.

To compute axisymmetric flow without swirl we can introduce the stream function ψ for the solenoidal velocity in cylindrical polar coordinates:

$$u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}. \tag{3.68}$$

This ψ is often referred to as *the Stokes stream function.* Then

$$\omega_\theta = -\frac{1}{r} L(\psi), \quad L \equiv \frac{\partial^2}{\partial z^2} + \frac{ptl^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}. \tag{3.69}$$

In the *steady* case, the vorticity equation gives

$$\Big[\frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial}{\partial z} - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial}{\partial f}\Big]\frac{1}{r^2} L(\psi) = 0. \tag{3.70}$$

Thus a family of steady solutions can be obtained by solving any equation of the form

$$L(\psi) = r^2 f(\psi), \tag{3.71}$$

where f is an arbitrary function, for the stream function ψ. The situation here is closely analogous to the steady two-dimensional case, see the previous subsection.

Now turning to axisymmetric flow *with* swirl, the instantaneous streamline and vortex lines can now be helices and a much larger class of Euler flows results. The same stream function applies. The swirl velocity satisfies, from (3.59)

$$\frac{Dru_\theta}{Dt} = 0, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_z\frac{\partial}{\partial z} + u_r\frac{\partial}{\partial r}. \tag{3.72}$$

We can understand the meaning of (3.72) using Kelvin's theorem. First note that a ring of fluid particles initially on a given circle C defined by initial values of z, r, will stay on the same circular ring as it evolves. The u_θ component takes the ring into itself, and the $(u_z, u_r, 0)$ sub-field determines the trajectory $C(t)$ of the ring, and thus the ring evolves as a material curve. Since u_θ is constant on the ring, the circulation on $C(t)$ is $2\pi r u_\theta$. By Kelvin's theorem, this circulation is a material invariant, and we obtain (3.72).

In the case of *steady* axisymmetic flow with swirl we see from (3.72) that we may take

$$ru_\theta = g(\psi), \tag{3.73}$$

where the function g is arbitrary. Bernoulli's theorem for steady flow with constant density gives

$$\frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} = H(\psi), \tag{3.74}$$

stating that the Bernoulli function H is constant on streamlines. From the momentum equation in the form $\nabla H - \mathbf{u}\times\boldsymbol{\omega} = 0$ we get, from the z-component e.g.:

$$u_r\omega_\theta - u_\theta\omega_r = \frac{\partial H}{\partial z}. \tag{3.75}$$

Using the expressions for the components of vorticity and expressing everything in terms of the stream function, we get from (3.73) and (3.75)

$$\frac{1}{r^2}\frac{\partial\psi}{\partial z} + \frac{1}{r}g\frac{dg}{d\psi}\frac{\partial\psi}{\partial z} = \frac{dH}{d\psi}\frac{\partial\psi}{\partial z}. \tag{3.76}$$

Eliminating the common factor $\frac{\partial\psi}{\partial z}$ and rearranging,

$$L(\psi) = r^2 f(\psi) - rg\frac{dg}{d\psi}, \quad f(\psi) = \frac{dH}{d\psi}. \tag{3.77}$$

Thus two arbitrary functions, f, g are involved and any solution of (3.77) determines a steady solution in axisymmetric flow with swirl.

Problem set 3

1. Consider a fluid of constant density in two dimensions with gravity, and suppose that the vorticity $u_y - v_x$ is everywhere constant and equal to ω. Show

that the velocity field has the form $(u, v) = (\phi_x + \chi_y, \phi_y - \chi_x)$ where ϕ is harmonic and χ is any function of x, y (independent of t), satisfying $\nabla^2\chi = -\omega$. Show further that

$$\nabla(\phi_t + \frac{1}{2}q^2 + \omega\psi + p/\rho + gz) = 0$$

where ψ is the stream function for $\mathbf{u}$, i.e. $\mathbf{u} = (\psi_y, -\psi_x)$, and $q^2 = u^2 + v^2$.

2. Show that, for an incompressible fluid, but one where the density can vary independently of pressure (e.g. salty seawater), the vorticity equation is

$$\frac{D\omega}{Dt} = \omega \cdot \nabla\mathbf{u} + \rho^{-2}\nabla\rho \times \nabla p.$$

Interpret the last term on the right physically. (e.g. what happens if lines of constant p are $y =$ constant and lines of constant ρ are $x - y =$constant?). Try to understand how the term acts as a source of vorticity, i.e. causes vorticity to be created in the flow.

3. For steady two-dimensional flow of a fluid of constant density, we have

$$\rho\mathbf{u} \cdot \nabla\mathbf{u} + \nabla p = 0, \nabla \cdot \mathbf{u} = 0.$$

Show that, if $\mathbf{u} = (\psi_y, -\psi_x)$, these equations imply

$$\nabla\psi \times \nabla(\nabla^2\psi) = 0.$$

Thus, show that a solution is obtained by giving a function $H(\psi)$ and then solving $\nabla^2\psi = H'(\psi)$. Show also that the pressure is given by $\frac{p}{\rho} = H(\psi) - \frac{1}{2}(\nabla\psi)^2$+constant.

4. Prove *Ertel's theorem* for a fluid of constant density: If f is a scalar material invariant, i.e. $Df/Dt = 0$, then $\omega \cdot \nabla f$ is also a material invariant, where $\omega = \nabla \times \mathbf{u}$ is the vorticity field.

5. A steady *Beltrami flow* is a velocity field $\mathbf{u}(\mathbf{x})$ for which the vorticity is always parallel to the velocity, i.e. $\nabla \times \mathbf{u} = f(\mathbf{x})\mathbf{u}$ for some scalar function f. Show that if a steady Beltrami field is also the steady velocity field of an inviscid fluid of constant density, the necessarily f is constant on streamlines. What is the corresponding pressure? Show that $\mathbf{u} = (Bsiny + C\cos z, C\sin z + A\cos x, A\sin x + B\cos y)$ is such a Beltrami field with $f = -1$. (This last flow an example of a velocity field yielding chaotic particle paths. This is typical of 3D Beltrami flows with constant f, according to a theorem of V. Arnold.)

6. Another formula exhibiting a vector field $\mathbf{u} = (u, v, w)$ whose curl is $\boldsymbol{\omega} = (\xi, \eta, \zeta)$, where $\nabla \cdot \boldsymbol{\omega} = 0$, is given by

$$u = z\int t\eta(tx, ty, tz)dt - y\int t\zeta(tx, ty, tz)dt,$$

$$v = x\int t\zeta(tx,ty,tz)dt - z\int t\xi(tx,ty,tz)dt,$$

$$w = y\int t\xi(tx,ty,tz)dt - x\int t\eta(tx,ty,tz)dt.$$

Verify this result. (Note that $\mathbf{u}$ will not in general be divergence-free, e.g. check $\xi = \zeta = 0, \eta = x$. A derivation of this formula, using differential forms, may be found in Flanders' book on the subject.)

7. In this problem the object is to find a 2D propagating vortex dipole structure analogous to that studied in subsection 3.6.2. In the present case, the structure will move clockwise on the circle of radius R with angular velocity Ω. Consider a rotating coordinate system and a circular structure of radius a, stationary and with center at $(0, R)$. Relative to the rotating system the velocity tends to $\Omega(-y, x) = \Omega(-y', x) + \Omega R(-1, 0), y' = y - R$. It turns out that (assuming constant density), the momentum equation relative to the rotating frame can be reduced to that in the non-rotating frame in that the Coriolis force can be absorbed into the gradient of a modified pressure, see a later chapter. Thus we again take $\nabla^2\psi + k^2\psi = 0, r' < a$. Here $r' = \sqrt{(y')^2 + x^2}$. A new term proportional to $J_0(kr)$ must now be included. We require that u_θ aAand ω must be continuous on $r' = a$. Show that, relative to the rotating frame,

$$\psi = \begin{cases} -\frac{2R\Omega}{kJ_0(ak)}\sin\theta J_1(kr') + \frac{2\Omega}{k^2 J_0(ka)}J_0(kr'), & \text{if } r' < a, \\ -\frac{\Omega}{2}r'^2 - \Omega R(r' - a^2/r') + \Omega a^2 \ln r' + C, & \text{if } r' \geq a. \end{cases} \tag{3.78}$$