Chapter 2

Conservation of mass and momentum

2.1 Conservation of mass

Every fluid we consider is endowed with a non-negative *density*, usually denoted by ρ, which is in the Eulerian setting is a scalar function of $\mathbf{x}, t$. Its unit are mass per unit volume. Water has a density of about 1 gram per cubic centimeter. For air the density is about 10^{-3} grams per cubic centimeter, but of course pressure and temperature affect air density significantly. The air in a room of a thousand cubic meters= 10^9 cubic centimeters weighs about a thousand kilograms, or more than a ton!

2.1.1 Eulerian form

Let us suppose that mass is being added or subtracted from space as a function $q(\mathbf{x}, t)$, of dimensions mass per unit volume per unit time. The conservation of mass in a fixed region $\mathcal{R}$ can be expressed using (1.20) with $f = \rho$:

$$\frac{d}{dt}\int_{\mathcal{R}} \rho dV_{\mathbf{x}} = \int_{\mathcal{R}} \frac{\partial \rho}{\partial t} dV_{\mathbf{x}} + \int_{\partial\mathcal{R}} \rho \mathbf{u} \cdot \mathbf{n} dS_{\mathbf{x}}. \tag{2.1}$$

Now

$$\frac{d}{dt}\int_{\mathcal{R}} \rho dV_{\mathbf{x}} = \int_{\mathcal{R}} q dV_{\mathbf{x}} \tag{2.2}$$

and if we bring the surface integral in (2.1) back into the volume integral using the divergence theorem we arrive at

$$\int_{\mathcal{R}} \Big[\frac{\partial \rho}{\partial t} + \mathrm{div}(\mathbf{u}\rho) - q\Big] dV_{\mathbf{x}} = 0. \tag{2.3}$$

Since our functions are continuous and $\mathcal{R}$ is an arbitrary open set in R_N, the integrand in (2.3) must vanish, yielding the conservation of mass equation in the Eulerian form:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\mathbf{u}\rho) = q. \tag{2.4}$$

Note that this last equation can also be written

$$\frac{D\rho}{Dt} + \rho \mathrm{div}\, \mathbf{u} = q. \tag{2.5}$$

The conservation of mass equation in either of these forms is sometimes called (for obscure reasons) the *equation of continuity*.

The form (2.5) shows that the material derivative of the density changes in two ways, either by sources and sinks of mass $q > 0$ or $q < 0$ respectively, or else by the non-vanishing of the divergence of the velocity field. A positive value of the divergence, as for $\mathbf{u} = (x, y, z)$, is associated with an expansive flow, thereby reducing local density. This can be examined more closely as follows. Let V be a small volume of fluid where the density is essentially constant. Then ρV is the mass within this fluid parcel, which is a material invariant $D(\rho V)/Dt = 0$. Thus $D\rho/Dt + \rho V^{-1} DV/Dt = 0$. Comparing this with (2.5) we have

$$\mathrm{div}\, \mathbf{u} = \frac{1}{V}\frac{DV}{Dt}. \tag{2.6}$$

Example 2.1: As we have seen in Chapter 1, an incompressible fluid satisfies div $\mathbf{u} = 0$. For such a fluid, free of sources or sinks of mass, we have

$$\frac{D\rho}{Dt} = 0, \tag{2.7}$$

that is, now density becomes a material property. This does not say that the density is constant everywhere in space, only that is constant at a given fluid parcel, as it moves about . (Note that we use parcel here to suggest that we have to average over a small volume to compute the density.) However a fluid of constant density without mass addition *must* be incompressible. This difference is important. Sea water is essentially incompressible but density changes due to salinity are an important part of the dynamics of the oceans.

2.1.2 Lagrangian form

If $q = 0$ the Lagrangian form of the conservation of mass is very simple because if we move with the fluid the density changes that we see are due to expansion and dilation of the fluid parcel, which is controlled by Det($\mathbf{J}$). Let a parcel have volume V_0 initially, with essentially constant initial density ρ_0. Then the mass of the parcel is $\rho_0 V_0$, and is a material invariant. At later times the density is ρ and the volume is $V_0\mathrm{Det}(\mathbf{J})$, so conservation of mass is expressed by

$$\mathrm{Det}\mathbf{J}(\mathbf{a}, t) = \frac{\rho_0}{\rho}. \tag{2.8}$$

If $q \neq 0$ the Lagrangian conservation of mass must be written

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{a}} \rho \mathrm{Det}(\mathbf{J}) = \mathrm{Det}(\mathbf{J}) q(\mathbf{x}(\mathbf{a}, t), t). \tag{2.9}$$

It is easy to get from Eulerian to Lagrangian form using (1.14). Assuming $q = 0$,

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{u} = 0 = \frac{D\rho}{Dt} + \rho \frac{D\mathrm{Det}(\mathbf{J})/Dt}{\mathrm{Det}(\mathbf{J})} = \frac{1}{\mathrm{Det}(\mathbf{J})}\frac{D}{Dt}(\rho \mathrm{Det}(\mathbf{J})) \quad (2.10)$$

and the connection is complete.

Example 2.2: Consider, in one dimension, the unsteady velocity field $u(x,t) = \frac{2xt}{1+t^2}$. We assume no sources of sinks of mass, and set $\rho(x,0) = x$. What is the density field at later times, in both Eulerian and Lagrangian forms? First note that this is a reasonable question, since we have a conservation of mass equation to evolve the density in time. First deriving the Lagrangian coordinates, we have

$$\frac{dx}{dt} = \frac{2xt}{1+t^2}, \quad x(0) = a. \quad (2.11)$$

The solution is $x = a(1+t^2)$. The Jacobian is then $J = 1 + t^2$. The equation of conservation of mass in Lagrangian form, given that $\rho_0(a) = a$, is $\rho = a/(1+t^2)$. Since $a = x/(1+t^2)$, the Eulerian form of the density is $\rho = x(1+t^2)^2$. It is easy to check that this last expression satisfies the Eulerian conservation of mass equation in one dimension $\rho_t + (\rho u)_x = 0$.

Example 2.3 Consider the two-dimensional stagnation-point flow $(u,v) = (x,-y)$ with initial density $\rho_0(x,y) = x^2 + y^2$ and $q = 0$. The flow is incompressible, so ρ is material. In Lagrangian form, $\rho(a,b,t) = a^2 + b^2$. To find ρ as a function of x, y, t, we note that the Lagrangian coordinates of the flow are $(x,y) = (ae^t, be^{-t})$, and so

$$\rho(x,y,t) = (xe^{-t})^2 + (ye^t)^2 = x^2e^{-2t} + y^2e^{2t}. \quad (2.12)$$

The lines of constant density, which are initially circles centered at the origin, are flattened into ellipses by the flow.

2.1.3 Another convection identity

Frequently fluid properties are most conveniently thought of as densities per unit mass rather than per unit volume. If the conservation such a quantity, f say, is to be examined, we will need to consider ρf to get "f per unit volume" and so be able to compute total amount by integration over a volume. Consider then

$$\frac{d}{dt}\int_{\mathcal{S}_t} \rho f dV_{\mathbf{x}} = \int_{\mathcal{S}_t} \left[\frac{\partial \rho f}{\partial t} + \operatorname{div}(\rho f \mathbf{u})\right] dV_{\mathbf{x}}. \quad (2.13)$$

We now assume conservation of mass with $q = 0$. From the product rule of differentiation we have $\operatorname{div}(\rho f \mathbf{u}) = f \operatorname{div}(\rho \mathbf{u}) + \rho \mathbf{u} \cdot \nabla f$, and so the integrand splits into a part which vanishes by conservation of mass, and a material derivative of f time the density:

$$\frac{d}{dt}\int_{\mathcal{S}_t} \rho f dV_{\mathbf{x}} = \int_{\mathcal{S}_t} \rho \frac{Df}{Dt} dV_{\mathbf{x}}. \quad (2.14)$$

Thus the effect of the multiplier ρ is to turn the derivative of the integral into an integral of a material derivative.

2.2 Conservation of momentum in an ideal fluid

The *momentum* of a fluid is defined to be $\rho\mathbf{u}$, per unit volume. Newton's second law of motion states that momentum is conserved by a mechanical system of masses if no forces act on the system. We are thus in a position to use (2.14), where the "sources and sinks" of momentum are *forces*.

If $\mathbf{F}(\mathbf{x}, t)$ is the force acting on the fluid, per unit volume, then we have immediately (assuming conservation of mass with $q = 0$),

$$\rho\frac{D\mathbf{u}}{Dt} = \mathbf{F}. \tag{2.15}$$

Since we have seen that $\frac{D\mathbf{u}}{Dt}$ is the fluid acceleration, (2.15) states Newton's Law that mass times acceleration equals force, in both magnitude and direction.

Of course the Lagrangian form of (2.15) is obtained by replacing the acceleration by its Lagrangian counterpart:

$$\rho\frac{\partial^2\mathbf{x}}{\partial t^2}\bigg|_{\mathbf{a}} = \mathbf{F}. \tag{2.16}$$

The main issues involved with conservation of momentum are those connected with the forces which are on a parcel of fluid. There are many possible forces to consider: pressure, gravity, viscous, surface tension, electromotive, etc. Each has a physical origin and a mathematical model with a supporting set of observation and analysis. In the present chapter we consider only an *ideal fluid*. The only new fluid variable we will need to introduce is the *pressure*, a scalar function $p(\mathbf{x}, t)$.

In general the force $\mathbf{F}$ appearing in (2.15) is assumed to take the form

$$F_i = f_i + \frac{\partial\sigma_{ij}}{\partial x_j}. \tag{2.17}$$

Here $\mathbf{f}$ is a body force (exerted from the "outside"), and σ is a second-order tensor called the *stress tensor*. Integrated over a region $\mathcal{R}$, the force on the region is

$$\int_{\mathcal{R}} \mathbf{F} dV_{\mathbf{x}} = \int_{\mathcal{R}} \mathbf{f} dV_{\mathbf{x}} + \int_{\partial\mathcal{R}} \sigma\cdot\mathbf{n} dS_{\mathbf{x}}, \tag{2.18}$$

using the divergence theorem. We can thus see that the effect of the stress tensor is to produce a force on the boundary of any fluid parcel, the contribution from an area element to this force being $\sigma_{ij}n_j dS_{\mathbf{x}}$ for an outward normal $\mathbf{n}$. The remaining body force $\mathbf{f}$ will sometimes be taken to be a uniform gravitational field $\mathbf{f} = \rho\mathbf{g}$, where $\mathbf{g} =$ constant. On the surface of the earth gravity acts toward the Earth's center with a strength $g \approx 980\ cm/sec^2$. We also introduce a general force potential Φ, such that $\mathbf{f} = -\rho\nabla\Phi$.

2.2.1 The pressure

An ideal fluid is defined by a stress tensor of the form

$$\sigma_{ij} = -p\delta_{ij} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}, \tag{2.19}$$

where $\delta_{ij} = i, i = j, = 0$ otherwise. Thus when pressure is positive the force on the surface of a parcel is opposite to the outer normal, as intuition suggests. Note that now

$$\text{div }\sigma = -\nabla p. \tag{2.20}$$

For a compressible fluid the pressure accounts physically for the resistance to compression. But pressure persists as a fundamental source of surface forces for an incompressible fluid, and its physical meaning in the incompressible case is subtle.[1]

An ideal fluid with no mass addition and no body force thus satisfies

$$\rho\frac{D\mathbf{u}}{Dt} + \nabla p = 0, \tag{2.21}$$

together with

$$\frac{D\rho}{Dt} + \rho\text{div }\mathbf{u} = 0. \tag{2.22}$$

This system of equation for an ideal fluid are also often referred to as *Euler's equations*. The term *Euler flow* is also in wide use.

With Euler's system we have $N + 1$ equations for the $N + 2$ unknowns $u_1, \ldots, u_N, \rho, p$. Another equation will be needed to complete the system. One possibility is the incompressible assumption div $\mathbf{u} = 0$. A common option is to assume constant density. Then ρ is eliminated as an unknown and the conservation of mass equation is replaced by the incompressibility condition. For gases the missing relation is an equation of state, which brings in the thermodynamic properties of the fluid.

The pressure force as we have defined it above is *isotropic*, in the sense the pressure is the same independently of the orientation of the area element on which it acts. A simple two-dimensional diagram will illustrate why this is so, see figure 2.1. Suppose that the pressure is p_i on the face of length L_i. Equating forces, we have $p_1 L_1 \cos\theta = p_2 L_2, p_1 L_1 \sin\theta = p_3 L_3$. But $L_1 \cos\theta = L_2, L_1 \sin\theta = L_3$, so we see that $p_1 = p_2 = p_3$. So indeed the pressure sensed by a face does not depend upon the orientation of the face.

2.2.2 Lagrangian form of conservation of momentum

The Lagrangian form of the acceleration has been noted above. The momentum equation of an ideal fluid requires that we express ∇p as a Lagrangian variable.

[1]One aspect of the incompressible case should be noted here, namely that the pressure is arbitrary up to an additive constant. Consequently it is only pressure *differences* which matter. This is not the case for a compressible gas.

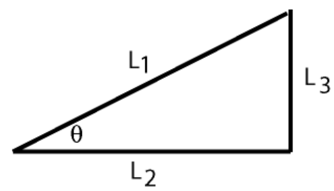

Figure 2.1: Isotropicity of pressure.

That is, if p is to be a function of $\mathbf{a}, t$ then since ∇ here is actually the $\mathbf{x}$ gradient $\nabla_{\mathbf{x}}$, we have $\nabla_{\mathbf{x}} p = \mathbf{J}^{-1}\nabla_{\mathbf{a}} p$. This appearance of the Jacobian is an awkward feature of Lagrangian fluid dynamics, and is one of the reasons that we shall emphasize Eulerian variables in discussing the dynamics of a fluid.

2.2.3 Hydrostatics: the Archimedean principle

Hydrostatics is concerned with fluids at rest ($\mathbf{u} = 0$), usually in the presence of gravity. We consider here only the case of a fluid stratified in one dimension. To fix the coordinates let the z-axis be vertical up, and $\mathbf{g} = -g\mathbf{i}_z$, where g is a positive constant. We suppose that the density is a function of z alone. This allows, for example, a body of water beneath a stratified atmosphere. Let a solid three-dimensional body (any deformation of a sphere for example) be submerged in the fluid. Archimedes principle says that the force exerted by the pressure on the surface of the body is equal to the total weight of the fluid displaced by the body. We want to establish this principle in the case considered.

Now the pressure satisfies $\nabla p = -g\rho(z)\mathbf{i}_z$. The pressure force is given by $\mathbf{F}_{pressure} = -\int p\mathbf{n}dS$ taken over the surface of the body. But this surface pressure is just the same as would be acting on a virtual surface within the fluid, no body present. Using the divergence theorem, we may convert this to an integral over the interior of this surface. Of course, there is no fluid within the body. We are just using the math to evaluate the surface integral. The result is $\mathbf{F}_{pressure} = g\mathbf{i}_z \int \rho dV$. This is a force upward equal to the weight of the displaced fluid, as stated.

2.3 Steady flow of a fluid of constant density

This special case gives us an opportunity to obtain some useful results rather easily in a class of problems of some importance. We shall allow a body force of the form $\mathbf{f} = -\rho\nabla\Phi$, so the momentum equation may be written, after division by the constant density,

$$\mathbf{u}\cdot\nabla\mathbf{u} + \rho^{-1}\nabla p + \nabla\Phi = 0. \tag{2.23}$$

We note now a vector identity which will be useful:

$$\mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{A} = \nabla(\mathbf{A}\cdot\mathbf{B}). \tag{2.24}$$

Applying this to $\mathbf{A} = \mathbf{B} = \mathbf{u}$ we have

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla |\mathbf{u}|^2 - \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{2.25}$$

Using (2.25) in (2.23) we have

$$\nabla(\rho^{-1} p + \Phi + \frac{1}{2}|\mathbf{u}|^2) = \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{2.26}$$

Taking the dot product with $\mathbf{u}$ on both sides we obtain

$$\mathbf{u} \cdot \nabla(\rho^{-1} p + \Phi + \frac{1}{2}|\mathbf{u}|^2) = 0. \tag{2.27}$$

The famous *Bernoulli theorem* for steady flows follows:*In the steady flow of an ideal fluid of constant density the quantity* $H \equiv \rho^{-1}p + \Phi + \frac{1}{2}|\mathbf{u}|^2$*, called the Bernoulli function, is constant on the streamlines of the flow.* The importance of this result is in the relation it gives us between velocity and pressure. Apart from the contribution of Φ, the constancy of H implies that an increase of velocity is accompanied by a decrease of the pressure. This is not an obvious dynamical consequence of the equations of motion, and it is interesting that we have derived it without referring to the solenoidal property of $\mathbf{u}$. Recall that the latter is implied by the constancy of density when there is no mass added or removed. If we make use of the solenoidal property then, using the identity $\nabla \cdot (\mathbf{A}\psi) = \psi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \phi$ for vector and scaler fields, we see that $\mathbf{u}H$ is also solenoidal, and so the flux of this quantity is conserved in stream tubes. This vector field arises when conservation of *mechanical energy*, relating changes in kinetic energy to the work done by forces, is studied, see problem 2.2.

It is helpful to apply the Bernoulli theorem to flow in a smooth rigid pipe of circular cross section and slowly varying diameter, with $\Phi = 0$. For an ideal (frictionless) fluid we may assume that the velocity is approximately constant over the section, this being reasonable if the slope of the wall of the pipe is small. The velocity may thus be taken as a scalar function $u(x)$. If the section area is $A(x)$, then the conservation of mass (and here, volume) implies that $uA \equiv Q =$ constant, so that $\rho^{-1}p + \frac{Q^2}{2}A^{-2} =$ constant. If we consider a contraction, as in figure 2.2., where the area and velocity go from A_1, u_1 to A_2, u_2, then the fluid speeds up to satisfy $A_1 u_1 = A_2 u_2 = Q$. To achieve this speedup in steady flow, a force must be acting on the fluid, here a pressure force. Conservation of momentum states the flux of momentum out minus the flux of momentum in must equal the pressure force on the fluid in the pipe between section 1 and section 2. Now $H = p/\rho + \frac{1}{2}(Q/A)^2$ is constant, so (if force is positive to the right) the two ends of the tube give a net pressure force $p_1 A_1 - p_2 A_2 = \rho Q^2/2(1/A_2 - 1/A_1)$ acting on the fluid. But there is also a pressure force along the curved part of the tube. This is seen to be $\int_{A_1}^{A_2} p dA = -\int_{A_1}^{A_2} \frac{\rho}{2}(Q/A)^2 dA = \rho Q^2/2(1/A_2 - 1/A_1)$. These two contributions are equal in our one-dimensional approximation, and their sum is $\rho Q^2(1/A_2 - 1/A_1)$ But the momentum out minus momentum in is $\rho(A_2 u_2^2 - A_1 u_1^2) = \rho Q^2(1/A_2 - 1/A_1)$ and is indeed

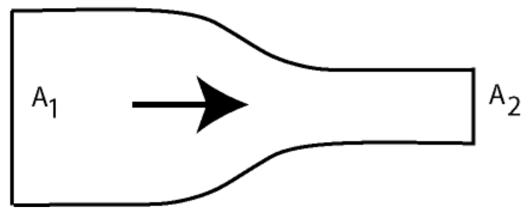

Figure 2.2: Steady flow through a contraction.

equal to the net pressure force. Intuitively then, to achieve the speedup of the fluid necessary to force the fluid through a contraction, and to maintain such a flow as steady in time, it is necessary to supply a larger pressure at station 1 than at station 2. Bernoulli's theorem captures this creation of momentum elegantly, but ultimately the physics comes down to pressure differences accelerating fluid parcels.

2.4 Intrinsic coordinates in steady flow

The one-dimensional analysis just given suggests looking briefly at the relations obtained in an arbitrary steady flow of an ideal fluid using the streamlines a part of the coordinate system. The resulting *intrinsic coordinates* are revealing of the dynamics of fluid parcels. Let $\mathbf{t}$ be the unit tangent vector to an oriented streamline. The we may write $\mathbf{u} = q\mathbf{t}, q = |\mathbf{u}|$. If s is arclength along the streamline, then

$$\frac{\partial \mathbf{u}}{\partial s} = \frac{\partial q}{\partial s}\mathbf{t} + q\frac{\partial \mathbf{t}}{\partial s} = \frac{\partial q}{\partial s}\mathbf{t} + q\kappa\mathbf{n}, \tag{2.28}$$

where $\mathbf{n}$ is the unit normal, κ the streamline curvature, and we have used the first Frenet-Serret formula. Now the operator $\mathbf{u}\cdot\nabla$ is just $q\frac{\partial}{\partial s}$, and so we have from (2.28)

$$\mathbf{u}\cdot\nabla\mathbf{u} = q\frac{\partial q}{\partial s}\mathbf{t} + q^2\kappa\mathbf{n}. \tag{2.29}$$

This shows that the acceleration in steady flow splits into a component along the streamline, determined by the variation of q, and a centripetal acceleration associated with streamline curvature. The equations of motion in intrinsic coordinates (zero body force) are therefore

$$\rho q\frac{\partial q}{\partial s} + \frac{\partial p}{\partial s} = 0, \quad \rho\kappa q^2 + \frac{\partial p}{\partial n} = 0. \tag{2.30}$$

What form does the solenoidal condition take in intrinsic coordinates? We consider this question in two dimensions. We have

$$\nabla \cdot \mathbf{u} = \nabla \cdot (q\mathbf{t}) = \mathbf{t} \cdot \nabla q + q\nabla \cdot \mathbf{t} = \frac{\partial q}{\partial s} + q\nabla \cdot \mathbf{t}. \tag{2.31}$$

Let us introduce an angle θ so that $\mathbf{t}(s) = (\cos\theta(s), \sin\theta(s))$. Then

$$\nabla \cdot \mathbf{t} = -\sin\theta \frac{\partial \theta}{\partial x} + \cos\theta \frac{\partial \theta}{\partial y} = \mathbf{n} \cdot \nabla \theta = \frac{\partial \theta}{\partial n}. \tag{2.32}$$

Since $\kappa = \frac{\partial \theta}{\partial s}$ is the streamline curvature, $\frac{\partial \theta}{\partial n}$, which we write as κ_n, is the curvature of the coordinate lines normal to the streamlines. Thus the solenoidal condition in two dimensions assumes the form

$$\frac{\partial q}{\partial s} + q\kappa_n = 0. \tag{2.33}$$

2.5 Potential flows with constant density

Another important and very large class of fluid flows are the so-called potential flows, defined as flows having a velocity field which is the gradient of a scalar *potential*, usually denoted by ϕ:

$$\mathbf{u} = \nabla \phi. \tag{2.34}$$

For simplicity we consider here only the case of constant density, but allow a body force $-\rho\nabla\Phi$ and permit the flow to be unsteady. Since we now also have that $\mathbf{u}$ is solenoidal, it follows that

$$\nabla \cdot \nabla \phi = \nabla^2 \phi = 0. \tag{2.35}$$

Thus the velocity field is determined by solving Laplace's equation (2.35)

The momentum equation has not yet been needed, but it necessary in order to determine the pressure, given $\mathbf{u}$. The momentum equation is

$$\mathbf{u}_t + \nabla(\frac{1}{2}|\mathbf{u}|^2 + p/\rho + \Phi) = \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{2.36}$$

Since $\mathbf{u} = \nabla\phi$ we now have $\nabla \times \mathbf{u} = 0$ and therefore

$$\nabla(\phi_t + \frac{1}{2}|\nabla\phi|^2 + p/\rho + \Phi) = 0, \tag{2.37}$$

or

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + p/\rho + \Phi = h(t). \tag{2.38}$$

The arbitrary function $h(t)$ may in fact be set equal to zero; otherwise we can replace ϕ by $\phi - \int h dt$ without affecting $\mathbf{u}$. We see that (2.38) is another "Bernoulli constant", this time applicable to any connected region of space where the potential flow is defined. It allows us to compute the pressure in an unsteady potential flow, see problem 2.6.

2.6 Boundary conditions on an ideal fluid

As we have noted, a main physical property of real fluid which is not present for an ideal fluid is a viscosity. The ideal fluid is"slippery", in the following sense. Suppose that adjacent to a solid wall the pressure varies along the wall. The only force a fluid parcel can experience is a pressure force associated with the pressure gradient. If the gradient at the wall is tangent to the wall, fluid will be accelerated and there will have to be a tangential component of velocity *at the wall.* This suggests that we cannot place any restriction on the tangential component of velocity at a rigid fixed boundary of the fluid.

On the other hand, by a rigid fixed wall we mean that fluid is unable to penetrate the wall, and so we will have to impose the condition $\mathbf{n}\cdot\mathbf{u} = u_n = 0$ on the wall. There is a subtlety here connected with our continuum approximation. It might be though that the fluid cannot penetrate *into* a rigid wall, but could it not be possible for the fluid to tear off the wall, forming a free interface next to an empty cavity? to see that this cannot be the case for smooth pressure fields, consider the reversed stagnation-point flow $(u, v) = (-x, y)$. On the upper y-axis we have a Bernoulli function $p/\rho + \frac{1}{2}y^2$. The gradient of pressure along this line is indeed accelerating the fluid away from the wall, but the fluid remains at rest at $x = y = 0$. We cannot really contemplate a pressure force on a particle, which might cause the particle to leave the wall, only on a parcel. In fact in this example fluid parcels near the y-axis are being compressed in the x-direction and stretched in the y-direction.

Thus, the appropriate boundary condition at a fixed rigid wall adjacent to an ideal fluid is

$$u_n = 0 \quad \text{on the wall.} \tag{2.39}$$

For a potential flow, this becomes

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on the wall.} \tag{2.40}$$

We shall find that these conditions at a rigid wall for an ideal fluid are sufficient to (usually uniquely) determine fluid flows in problems of practical importance.

Another way to express the appropriate boundary condition on a ideal fluid at a rigid wall is that *fluid particles on a wall stay on the wall.* This alternative is attractive because it is also true of a *moving* rigid wall, where the velocity component normal to the wall need not vanish at the wall. So what is the appropriate condition on a moving wall? To obtain this it is convenient to define the surface as a function of time by the equation $\Sigma(\mathbf{x}, t) = 0$. For a particle at position $\mathbf{x}_p(t)$ to be on the surface means that $\Sigma(\mathbf{x}_p(t), t) = 0$. Differentiating this expression with respect to time we obtain

$$\left.\frac{\partial \Sigma}{\partial t}\right|_{\mathbf{x}} + \mathbf{u}\cdot\nabla\Sigma = 0. \tag{2.41}$$

For example, let a rigid cylinder of radius a move in the x-direction with velocity U. Then $\Sigma = (x - Ut)^2 + y^2 - a^2$, and (**??**) becomes $-2U(x - Ut) + 2(x -$

$Ut)u + 2yv = 0$ Evaluating this on the surface of the cylinder, we get

$$u\cos\theta + v\sin\theta = U\cos\theta = u_n. \tag{2.42}$$

We remark that the same reasoning can be applied to the moving *interface* between two fluids. This interface may also be regarded as consisting of fluid particles that remain on the interface. We refer to this generalized boundary condition at a moving surface as a *surface condition*.

Finally, as part of this first look at the boundary condition of fluid dynamics, we should note that for unsteady fluid flows we will sometimes need to prescribe *initial conditions*, insuring that the fluid equations may be used to carry the solution forward in time.

Example 2.4: We consider an example of potential flow past a body in two dimensions, constant density, no body force. The body is the circular cylinder $r = a$, and the fluid " at infinity" has fixed velocity $(U, 0)$ In two dimensional polar coordinates, Laplace's equation has solutions of the form $\ln r, (r^n, r^{-n})(\cos\theta, \sin\theta),\ n = 1, 2, \ldots$. The potential $Ur\cos\theta = Ux$ has the correct behavior at infinity, and so we need a decaying solution which will insure the boundary condition $\frac{\partial\phi}{\partial r} = 0$ when $r = a$. The correct choice is clearly a multiple of $r^{-1}\cos\theta$ and we obtain

$$\phi = U\cos\theta(r + a^2/r) \tag{2.43}$$

Note that $U\cos a^2/r$ is the potential of a flow seen by an observer at rest relative to the fluid at infinity, when the cylinder moves relative to the fluid with a velocity $(-U, 0)$. We see that indeed this potential satisfies $\frac{\partial\phi}{\partial r}\big|_{r=a} = -U\cos\theta$ as required by (2.42). Streamlines both inside and outside the cylinder are shown in figure 2.3.

We have found a solution representing the desired flow, but is the solution unique? Perhaps surprisingly, the answer is no. The reason, associated with the fluid region being non-simply connected, will be discussed in chapter 4.

Example 2.5 An interesting case of unsteady potential flow occurs with deep water waves (constant density). The fluid at rest is a liquid in the domain $z < 0$ of R^3. Gravity acts downward so $\Phi = -gz$. The space above is taken as having no density and a uniform pressure p_0. If the water is disturbed, waves can form on the surface, which we will assume to be described by a function $z = Z(x, y, t)$ (no breaking of waves). Under appropriate initial conditions it turns out that we may assume the liquid velocity to be a potential flow. Thus our mathematical problem is to solve Laplace's equation in $z < Z(x, y, t)$ with a surface condition on ϕ and a pressure condition $p_{z=Z} = p_0$. For the latter we can use the Bernoulli theorem for unsteady potential flows, to obtain

$$p_0/\rho = \Big[-\phi_t - \frac{1}{2}|\nabla\phi|^2 + gz \Big]_{z=Z}. \tag{2.44}$$

The surface condition is $\frac{D}{Dt}(z - Z(x, y, t)) = 0$ or

$$\Big[z - Z_t - uZ_x - vZ_y \Big]_{z=Z} = 0. \tag{2.45}$$

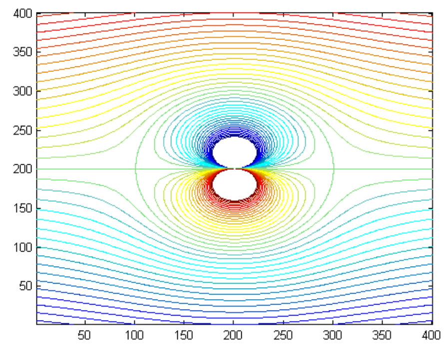

Figure 2.3: Potential flow past a circular cylinder.

The object is to find $\phi(x, y, z, t), Z(x, y, t)$, given e.g. that the water is initially at rest and that the fluid surface is at an initial elevation $z = Z_0(x, y)$. We will consider water waves in more detail in Chapter 9.

Problem set 2

1. For potential flow over a circular cylinder as discussed in class, with pressure equal to the constant p_∞ at infinity , find the static pressure on surface of the cylinder as a function of angle from the front stagnation point. (Use Bernoulli's theorem.) Evaluate the drag force (the force in the direction of the flow at infinite which acts on the cylinder), by integrating the pressure around the boundary. Verify that the drag force vanishes. This is an instance of *D'Alembert's paradox*, the vanishing of drag of bodies in steady potential flow.

2. For an ideal inviscid fluid of constant density, no gravity, the conservation of mechanical energy is studied by evaluating the time derivative of total kinetic energy in the form

$$\frac{d}{dt}\int_D \frac{1}{2}\rho|\mathbf{u}|^2 dV = \int_{\partial D} \mathbf{F}\cdot\mathbf{n}dS.$$

Here D is an arbitrary fixed domain with smooth boundary ∂D. What is the vector $\mathbf{F}$? Interpret the terms of $\mathbf{F}$ physically.

3. An open rectangular vessel of water is allowed to slide freely down a smooth frictionless plane inclined at an angle α to the horizontal, in a uniform vertical gravitational field of strength g. Find the inclination to the horizontal of the free surface of the water, given that it is a surface of constant pressure. We assume the fluid is at rest relative to an observer riding on the vessel. (Consider the acceleration of the fluid particles in the water and balance this against the gradient of pressure.)

4. Water (constant density) is to be pumped up a hill (gravity = $(0, 0, -g)$) through a pipe which tapers from an area A_1 at the low point to the smaller area A_2 at a point a vertical distance L higher. What is the pressure p_1 at the bottom, needed to pump at a volume rate Q if the pressure at the top is the atmospheric value p_0? (Express in terms of the given quantities. Assuming inviscid steady flow, use Bernoulli's theorem with gravity and conservation of mass. Assume that the flow velocity is uniform across the tube in computing fluid flux and pressure.)

5. For a *barotropic fluid*, pressure is a function of density alone, $p = p(\rho)$. In this case derive the appropriate form of Bernoulli's theorem for steady flow without gravity. If $p = k\rho^\gamma$ where γ, k are positive constants, show that $q^2 + \frac{2\gamma}{\gamma-1}\frac{p}{\rho}$ is constant on a streamline, where $q = |\mathbf{u}|$ is the speed.

6. Water fills a truncated cone as shown in the figure. Gravity acts down (the direction $-z$). The pressure at the top surface, of area A_2 is zero. The height of the container is H. At $t = 0$ the bottom, of area $A_1 < A_2$, is abruptly removed and the water begins to fall out. Note that at time $t = 0+$ the pressure at the bottom surface is also zero. The water has not moved but the acceleration is non-zero. We may assume the resulting motion is a potential flow. Thus the potential

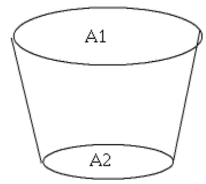

Figure 2.4: Truncated cone of fluid

$\phi(z, r, t)$ in cylindrical polars has the Taylor series $\phi(r, z, t) = t\Phi(r, z) + O(t^2)$, so $d\phi/dt = \Phi(r, z) + O(t)$. Using these facts, set up a mathematical problem for determining the pressure on the inside surface of cone at $t = 0+$. You should specify all boundary conditions. You do not have to solve the resulting problem, but can you guess what the surfaces Φ =constant would look like qualitatively? What is the force felt at $t = 0+$ by someone holding the cone, in the limits $A_1 \rightarrow 0$ and $A_1 \rightarrow A_2$?