INERTIAL SWIMMING AS A SINGULAR PERTURBATION

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ABSTRACT

The swimming of a sheet, originally treated by G.I. Taylor (1951) for the case of Stokes flow, is considered at moderate and high Reynolds numbers using matched asymptotic expansions. It is shown that for propagating waves with frequency ω, wavenumber k, and amplitude b, the swimming speed must be deduced from a dual expansion in powers of the small parameters $bkR^{1/2}$ and $R^{-1/2}$, where $R = \frac{\omega}{\nu k^2}$ is the Reynolds number. The result of Tuck (1968) for the leading term of the swimming velocity is recovered, and higher-order results are given. For the case of a planar, stretching sheet, the expansion is in powers of bk and $R^{-1/2}$ and a limit for large R is obtained as a boundary layer. We contrast these results with the inviscid case, where no swimming is possible. We also consider briefly the application of these ideas to "recoil swimming", wherein the movements of the center of mass and center of volume of a body allow swimming at both finite and infinite Reynolds numbers.

INTRODUCTION

In his seminal paper on the swimming of micro-organisms [1], G.I. Taylor showed how waves on a two-dimensional sheet could cause the sheet to move, i.e. "swim", relative to the fluid at infinity. Taylor assumed the fluid dynamics of Stokes flow, but his results were extended by Tuck [2] to arbitrary finite Reynolds number. For the case of a progressive wave of amplitude b, wave number k, and phase velocity c, Tuck obtained, by expansion in the small parameter bk, the leading term

$$U_s = -\frac{1}{2}c(kb)^2\frac{1+F(R)}{2F(R)} \tag{1}$$

for the swimming velocity, where $F = \Re[\sqrt{1+iR}] = \left[\frac{1+(1+R^2)^{1/2}}{2}\right]^{1/2}$ and $R = \frac{\omega}{\nu k^2}$ is a Reynolds number based on the properties of the wave. This swimming velocity has a finite limit $-\frac{1}{4}c(kb)^2$ as $R \to \infty$. The rate of dissipation per unit area of sheet increases with R as $\sqrt{R}$, and in fact the velocity near the sheet is of order $bk\sqrt{R}$ for large R, so it is clear that Tuck's result should be regarded as asymptotic in bk for fixed R, and not uniformly valid in the limit of large R. The purpose of the present paper is to attempt to clarify the nature of the approximation at large R using the techniques of matched asynmptotic expansions. We apply this approach to various cases considered by Taylor, and also consider analogous results for a perfect fluid.

FORMULATION

Since we shall be dealing with a nonlinear problem, we restrict attention to two of the problems studied by Taylor [1]. The first (I) is a progressive wave of shape on a two-dimensional elastic sheet, the equation of the boundary being $y = y_s(x) = b\sin(kx-\omega t), x_s = x,\ bk \ll 1$, each point on the sheet oscillating in the vertical. The second (II) is a progressive wave of stretching on a flat sheet. Here the Lagrangian coordinates of a point on the sheet with initial position $x_0 + a\sin(kx_0), y_0 = 0,\ ak \ll 1$ are given by $x_s == x_0 + \cos(kx_0 - \omega t), y_s = 0$. The streamfunction for the fluid flow, $\psi(x,y,t)$, $(u,v) = (\psi_y, -\psi_x)$ being the velocity field, satisfies the two-dimensional Navier-Stokes equation

$$\frac{\partial\nabla^2\psi}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \nu\nabla^4\psi = 0. \tag{2}$$

The fluid occupies the domain above the surface of the sheet. We assume tha ψ_x and ψ_{yy} vanish as $y \to \infty$, i.e. the flow is at most a uniform flow in the x-direction at $y = +\infty$. On the sheet surface the no-slip condition prevails:

$$u(x_s, y_s, t) = 0, v(x_s, y_s, t) = -b\omega\cos(kx_s - \omega t) \quad \text{(I)} \tag{3}$$

at a point (x_s, y_s), where $y_s = b\sin(kx_s - \omega t), x_s = x$ for problem I. For problem II

$$u(x_s, y_s, t) = a\omega\sin(kx_s - \omega t), v(x_s, y_s, t) = 0 \quad \text{(II)} \tag{4}$$

with $x_s = x + a\cos(kx_s - \omega t) = 0, y_s = 0$ and $|ak| < 1$. Passing to the dimensionless variables $(\xi, \eta) = (kx - \omega t, ky), \Psi(\xi, \eta) = (k^2/\omega)\psi(x, y, t)$ we have the following problem in ξ, η:

$$-\frac{\partial \nabla^2 \Psi}{\partial \xi} + \frac{\partial \Psi}{\partial \eta}\frac{\partial \nabla^2 \Psi}{\partial \xi} - \frac{\partial \Psi}{\partial \xi}\frac{\partial \nabla^2 \psi}{\partial \eta} - \frac{1}{R}\nabla^4 \Psi = 0, \tag{5}$$

$$\Psi_\xi(\xi, \varepsilon\sin\xi) = \varepsilon\cos\xi,\ \Psi_\eta(\xi, \varepsilon\sin\xi) = 0 \ \text{(I)}, \tag{6}$$

$$\Psi_\xi(\xi + \varepsilon\cos\xi, 0) = 0,\ \Psi_\eta(\xi + \varepsilon\cos\xi, 0) = \varepsilon\sin\xi \quad \text{(II)}. \tag{7}$$

Here $\nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}, \varepsilon = bk$ (I), $= ak$ (II), and $R = \frac{\omega}{\nu k^2}$ is the Reynolds number. In our analysis ε will be small and the exact problem will be replaced by the construction of asymptotic expansions defined on the half-space $\eta > 0$.

The direct calculation for I

To indicate the approach of Tuck [2] for problem I, we expand in ε:

$$\Psi(\xi, \eta; \varepsilon, R) = \varepsilon\Re[\Psi_1(\eta; R)e^{-i\xi}]$$

$$+\varepsilon^2\Psi_{20}(\eta, R) + \varepsilon^2\Re[\Psi_{22}(\eta, R)e^{-2i\xi}] + O(\varepsilon^3). \tag{8}$$

Here Ψ_1 is the leading solution of a linear problem, Ψ_{22} is the zero-mean component of the second-order term generated by the nonlinearities, and Ψ_{20} is a real-valued average over ξ of the second-order term. The negative of the swimming velocity is equal to the limit of $\frac{\partial \Psi_{20}}{\partial \eta}$ for large η. The nonlinearities come from two sources, the first being the inertial terms of (5), the second, restricted to problem I, being the displacement $\varepsilon\sin\xi$ of the boundary from the horizontal.

Now Ψ_1 satisfies the following linear problem:

$$i(D^2 - 1)\Psi_1 - \frac{1}{R}(D^2 - 1)^2\Psi_1 = 0, \tag{9}$$

$$\Psi(\xi, 0) = i,\ \frac{d\Psi_1}{d\eta}(\xi, 0) = 0, \tag{10}$$

where $D = d/d\eta$. The solution is

$$\Psi_1 = \frac{1+f}{R}[fe^{-\eta} - e^{-f\eta}],\ f = (1 + iR)^{1/2}. \tag{11}$$

We note that, after substituting Ψ at this order into (5), Ψ_{20} is found to satisfy

$$\frac{1}{R}D^4\Psi_{20} = -\frac{1}{2}\langle|1+f|^2\Re[f(1+f^*)e^{-(1+f^*)\eta}$$

$$-(f+f^*)e^-(f+f^*)\eta]\rangle, \tag{12}$$

where f^* is the complex conjugate of f and $\langle\cdot\rangle$ denotes the ξ-average. The second source of nonlinearity comes into play in the boundary conditions on Ψ_{20}:

$$D\Psi_{20}(0) = -\langle\sin\xi\Re[e^{-\xi}D^2\Psi_1]\rangle,\ D^2\Psi_{20}(\infty) = 0. \tag{13}$$

Integration of (12) and the boundary conditions then yields

$$D\Psi_{20}(\infty) = -U_s = \frac{1}{4}(f+f^*) + \frac{1}{2}|1+f|^2$$

$$\times\Re\Big[\frac{f}{(1+f^*)^2} - \frac{1}{f+f^*}\Big]. \tag{14}$$

Some manipulation of this last expression using $f^2 + f^{*2} = 2$ and $ff^* = 2F^2 - 1$ yields (1).

It should be noted from (11) that when R is large the term $e^{-f\eta}$ decays like $e^{-\eta\sqrt{R/2}}$, indicating a boundary-layer structure of thickness $O(R^{-1/2})$ relative to the wavelength.

Analysis of I for large R using matched expansions

We now consider simultaneously the two assumptions of small ε and large R. The boundary-layer structure referred to above suggests that the direct perturbational approach has implicitly assumed that $\varepsilon R^{1/2} \ll 1$. We shall now verify this and introduce asymptotic expansions in powers of $\delta \equiv \varepsilon R^{1/2}$ and $R^{-1/2}$.

The inner expansion

To study the structure of the flow near the sheet at large R it is natural to introduce the *inner variables*

$$\overline{\Psi} = R^{1/2}\Psi,\ \overline{\eta} = R^{1/2}\eta. \tag{15}$$

If (5) is expressed in these inner variables, the $\frac{1}{R}$ factor of the viscous stress term is expelled from the leading term and $\nabla^2\Psi$ is replaced by $R^{1/2}(\overline{D}^2\overline{\Psi} - R^{-1}\overline{\Psi}_{\xi\xi})$. Taking the limit as $R \to \infty$ in these variables yields the *inner limit* of (5):

$$-\overline{D}^2\overline{\Psi}_\xi + \overline{D\Psi D}^2\Psi_\xi - \overline{\Psi}_\xi\overline{D}^3\overline{\Psi} - \overline{D}^4\overline{\Psi} = 0. \tag{16}$$

Here $\overline{D} = \frac{\partial}{\partial\overline{\eta}}$. The conditions at the sheet for the inner limit of I are

$$\overline{\Psi}_\xi = \delta\cos\xi,\ \overline{\Psi}_{\overline{\eta}} = 0,\ \overline{\eta} = \delta\sin\xi. \tag{17}$$

Thus only δ survives as a parameter; this new expansion parameter is a measure of the wave amplitude relative to the boundary-layer thickness. The inner limit may thus be constructed as an expansion for small δ, and the inner expansion may be developed as a double expansion in δ and $R^{-1/2}$. In the representation of these terms it will be helpful to assign an ordering of $R^{-1/2}$ equal to $O(\delta)$, so that Nth-order terms will will be of order $\delta^m R^{-n/2}$ with $m+n = N$. The corresponding term of the inner expansion will be $\overline{\Psi}_{mn}$. Note the subscripts used in (8) have a different meaning. Note also that we must allow for a possible but hidden dependence of every term on the two parameters, corresponding to terms of order intermediate between powers, often consisting of logarithmic factors. In fact for the terms considered below we will not encounter these intermediate orders.

The inner expansion thus takes the form

$$\overline{\Psi} \sim \overline{\Psi}_{10}\delta + \delta R^{-1/2}\overline{\Psi}_{11} + \left[\delta^2 R^{-1/2}\overline{\Psi}_{21} + \delta R^{-1}\overline{\Psi}_{12}\right]$$

$$+\left[\delta^2 R^{-1}\overline{\Psi}_{22} + \delta R^{-3/2}\overline{\Psi}_{13} + \delta^3 R^{-1/2}\overline{\Psi}_{31}\right] + O(\delta^5). \tag{18}$$

Note that there is no term $\overline{\Psi}_{01}$, nor terms $\overline{\Psi}_{0n}$ or $\overline{\Psi}_{n0}$ for $n > 1$. The leading term is

$$\overline{\Psi}_{10} = \sin\xi. \tag{19}$$

The outer expansion

As in conventional boundary-layer theory, we postulate the outer expansion as a series of harmonic functions, to which the boundary-layer structure must decay, and with which it must match. The outer expansion will have a form similar to the inner expansion:

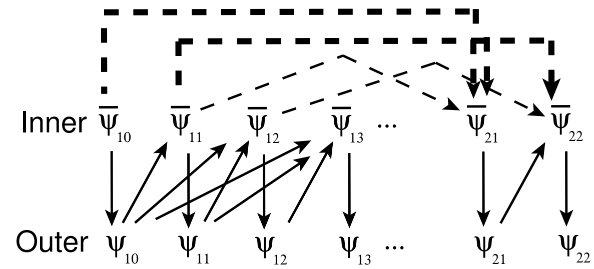

Figure 1. Terms of the inner and outer expansions for problem I needed to compute the leading term for swimming velocity. Heavy dashed line indicated forcing by nonlinearity of the equations. Light dashed lines indicate forcing through the boundary conditions. Solid lines indicate matching conditions are determining.

$$R^{1/2}\Psi \sim \Psi_{10}\delta + \delta R^{-1/2}\Psi_{11} + \left[\delta^2 R^{-1/2}\Psi_{21} + \delta R^{-1}\Psi_{12}\right]$$

$$+\left[\delta^2 R^{-1}\Psi_{22} + \delta R^{-3/2}\Psi_{13} + \delta^3 R^{-1/2}\Psi_{31}\right] + O(\delta^5). \tag{20}$$

Note the factor of $R^{1/2}$ in the left. The term Ψ_{10} matches with $\overline{\Psi}_{10}$, in the sense that

$$\lim_{\overline{\eta}\to\infty} \overline{\Psi}_{10} = \lim_{\eta\to 0} \Psi \tag{21}$$

In general, however, matching must occur in an intermediate overlap region where limits are taken with a variable of the form $\eta_\alpha = R^\alpha\eta, 0 < \alpha < 1/2$ held fixed [3].

We therefore have $\Psi_{10} = \sin\xi e^{-\eta}$. The inner expansion of Ψ_{10} is $\sin\xi[1 - R^{-1/2}\overline{\eta} + O(R^{-1})]$. The $O(R^{-1/2})$ part matches with $\overline{\Psi}_{11}$.

In preparation for the derivation of subsequent terms, we show in figure 1 a matching a diagram illustrating the relations between terms of the inner and outer expansions.

$\overline{\Psi}_{11}$ and Ψ_{11}

Now $\overline{\Psi}_{11}$ satisfies the following homogeneous linear problem:

$$\overline{D}^2\partial_\xi\overline{\Psi}_{11} + \overline{D}^4\overline{\Psi}_{11} \equiv L\overline{\Psi}_{11} = 0, \tag{22}$$

$$\overline{\Psi}_{11\xi}(\xi, 0) = \overline{D\Psi}_{11}(\xi, 0) = 0. \tag{23}$$

The solution, behaving like $-\overline{\eta}\sin\xi$ for large $\overline{\eta}$, is

$$\overline{\Psi}_{11} = -\overline{\eta}\sin\xi - \frac{1}{\sqrt{2}}e^{-Y}(\sin X + \cos X) + \frac{1}{\sqrt{2}}(\sin\xi + \cos\xi). \tag{24}$$

Here and elsewhere we will use the notation

$$Y = \overline{\eta}/\sqrt{2},\ X = \xi + \overline{\eta}/\sqrt{2}. \tag{25}$$

We can thus see the origin of Ψ_{11}, to match with the terms $\frac{1}{\sqrt{2}}(\sin\xi + \cos\xi)$ in (24):

$$\Psi_{11} = \frac{1}{\sqrt{2}}(\sin\xi + \cos\xi)e^{-\eta}. \tag{26}$$

The 21 and 12 terms

The term $\overline{\Psi}21$ comes from the advection of $\overline{\Psi}_{11}$ by $\overline{\Psi}_{10}$, and satisfies

$$L\overline{\Psi}_{21} = -\cos\xi\,\overline{D}^3\overline{\Psi}_{11}, \tag{27}$$

with

$$\overline{\Psi}_{21\xi} + \overline{D}\partial_\xi\overline{\Psi}_{11}\ \sin\xi = 0,\ \overline{D\Psi}_{21} + \overline{D}^2\overline{\Psi}_{11}\ \sin\xi = 0 \tag{28}$$

when $\overline{\eta} = 0$. Integration and the conditions at infinity yield

$$\partial_\xi\overline{\Psi}_{21} + \overline{D}^2\overline{\Psi}_{21} = -\cos\xi\,\overline{D\Psi}_{11}. \tag{29}$$

We then obtain

$$\overline{\Psi}_{21} = -\frac{1}{2}\cos 2\xi - \sin\xi\, e^{-Y}\sin X. \tag{30}$$

Matching then yields

$$\Psi_{21} = -\frac{1}{2}\cos 2\xi\, e^{-2\eta}. \tag{31}$$

The term $\overline{\Psi}_{12}$ is forced by an $O(R^{-1})$ term in the equation in inner variables, satisfying

$$L\overline{\Psi}_{12} = -\partial_\xi^3\overline{\Psi}_{10} = \cos\xi, \tag{32}$$

together with homogeneous conditions on $\overline{\eta} = 0$, and matching with the inner expansions of Ψ_{10} and Ψ_{11} . It is given by

$$\overline{\Psi}_{12} = \frac{1}{2}\eta^2\sin\xi - \frac{\overline{\eta}}{\sqrt{2}}(\sin\xi + \cos\xi) - e^{-Y}\cos X + \cos\xi. \tag{33}$$

Matching then gives us the outer term

$$\Psi_{12} = \cos\xi\, e^{-\eta}. \tag{34}$$

The 31 and 13 terms

Similarly, $\overline{\Psi}_{31}$ solves

$$L\overline{\Psi}_{31} = -\cos\xi\,\overline{D}^3\overline{\Psi}_{21}, \tag{35}$$

with

$$\overline{D\Psi}_{31} = -\frac{1}{2}\overline{D}^3\overline{\Psi}_{11}\ \sin^2\xi - \overline{D}^2\overline{\Psi}_{21}\ \sin\xi \tag{36}$$

and

$$\partial_\xi\overline{\Psi}_{31} = -\frac{1}{2}\overline{D}^2\partial_\xi\overline{\Psi}_{11}\ \sin^2\xi - \overline{D}\partial_\xi\overline{\Psi}_{21}\ \sin\xi \tag{37}$$

on $\overline{\eta} = 0$. A particular solution of (35) is

$$\overline{\Psi}^p_{31} = \frac{1}{4\sqrt{2}}\cos 2\xi\, e^{-Y}(\sin X - \cos X). \tag{38}$$

We note that

$$D\overline{\Psi}^p_{31}|_{\overline{\eta}=0} = \frac{1}{4}\cos\xi\ \cos 2\xi = \frac{1}{8}(\cos 3\xi + \cos\xi), \tag{39}$$

$$\partial_\xi\overline{\Psi}^p_{31}|_{\overline{\eta}=0} = \frac{3}{8\sqrt{2}}(\sin 3\xi + \cos 3\xi) + \frac{1}{8\sqrt{2}}(\sin\xi - \cos\xi). \tag{40}$$

On the other hand the right-hand side of (36) equals $\frac{1}{8}(\cos 3\xi - \cos\xi)$, and that of (37) equals $\frac{3}{8\sqrt{2}}(\sin 3\xi + \cos 3\xi) - \frac{1}{8\sqrt{2}}(\sin\xi + 3\cos\xi)$. Thus we have, adding the requisite solution of the homogeneous equation,

$$\overline{\Psi}_{31} = \overline{\Psi}^p_{31} + \frac{1}{4\sqrt{2}}e^{-Y}(\cos X - \sin X). \tag{41}$$

The term $\overline{\Psi}_{13}$ is required to match with the appropriate terms of the inner expansions of Ψ_{1j}, $j = 0, 1, 2$, and is forced by linear terms involving $\overline{\Psi}_{11}$. The equation is

$$L\overline{\Psi}_{13} = -(\partial_\xi + 2\overline{D}^2)\partial_\xi^2\overline{\Psi}_{11}. \tag{42}$$

The boundary conditions are homogeneous. A particular solution is

$$\overline{\Psi}^p_{13} = \frac{1}{2}\overline{\eta}e^{-Y}\cos X + \frac{1}{2\sqrt{2}}\overline{\eta}^2(\sin\xi + \cos\xi) - \frac{1}{6}\overline{\eta}^3\sin\xi, \tag{43}$$

and to satisfy the condition on $\overline{D}\overline{\Psi}_{13}(\xi,0)$ as well as match with Ψ_{12} we have

$$\overline{\Psi}_{13} = \overline{\Psi}_{13}^{p} + \frac{1}{2\sqrt{2}}e^{-Y}(\sin X - \cos X) - \overline{\eta}\cos\xi - \frac{1}{2\sqrt{2}}(\sin\xi - \cos\xi). \tag{44}$$

Thus

$$\Psi_{31} = -\frac{1}{2\sqrt{2}}(\sin\xi - \cos\xi)e^{-\eta}. \tag{45}$$

It follows by matching that $\Psi_{31} = 0$.

$\overline{\Psi}_{22}$ and the swimming velocity

So far we have computed all terms with null conditions on $\Psi_{\overline{\eta}}$ at infinity, indicating that the sheet does not swim to the orders considered. We now compute the swimming velocity by consideration of the calculation of $\overline{\Psi}_{22}$. This term is forced by $\overline{\Psi}_{10}, \overline{\Psi}_{11}$ and $\overline{\Psi}_{12}$:

$$L\overline{\Psi}_{22} = \overline{D}\left[\overline{D}\overline{\Psi}_{11}\overline{D}\partial_\xi\overline{D}_{11} - \partial_\xi\overline{\Psi}_{11}\overline{D}^2\overline{\Psi}_{11}\right] - \partial_\xi\overline{\Psi}_{10}\overline{D}^3\overline{\Psi}_{12}. \tag{46}$$

There is a homogeneous condition on $\partial_\xi\overline{\Psi}_{22}$ on $\overline{\eta} = 0$, but

$$\overline{D}\overline{\Psi}_{22}\big|_{\overline{\eta}=0} = -\sin\xi\,\overline{D}^2\overline{\Psi}_{12}\big|_{\overline{\eta}=0}. \tag{47}$$

We will compute the swimming velocity at this order as in [1,2], by averaging over ξ and finding $\langle\overline{D}\overline{\Psi}_{22}\rangle(\infty) \equiv U_{22}$. The dimensionless swimming speed to leading order is therefore $U_s = -\delta^2 R^{-1}U_{22} = -\varepsilon^2 U_{22}$.

Averaging and integrating once in $\overline{\eta}$ we obtain

$$\overline{D}^3\langle\overline{\Psi}_{22}\rangle = \langle -\frac{1}{\sqrt{2}}(\sin\xi + cos\xi) + \overline{\eta}\sin\xi + E)E + \cos\xi\, e^{-Y}\sin X\rangle, \tag{48}$$

where $E = \frac{1}{\sqrt{2}}e^{-Y}(\sin + \cos X)$. Thus

$$\overline{D}^3\langle\overline{\Psi}_{22}\rangle = \frac{1}{2}[e^{-\sqrt{2}\overline{\eta}} - e^{-Y}\cos Y]$$

$$+\frac{1}{2\sqrt{2}}\overline{\eta}e^{-Y}(\cos Y - \sin Y) + \frac{1}{2}e^{-Y}\sin Y. \tag{49}$$

Integrating again from infinity,

$$\overline{D}^2\langle\overline{\Psi}_{22}\rangle = -\frac{1}{2\sqrt{2}}e^{-\sqrt{2}\overline{\eta}} + \frac{1}{2\sqrt{2}}e^{-Y}(\cos Y - \sin Y) + \frac{1}{2}\overline{\eta}e^{-Y}\sin Y. \tag{50}$$

Integrating (50) from 0 to ∞ and using (47) and the fact that $\langle\sin\xi\,\overline{D}^2\overline{\Psi}_{12}|_{\overline{\eta}=0}\rangle = 0$ we obtain

$$U_{22} = -\frac{1}{4} + 0 + \frac{1}{2} = \frac{1}{4}. \tag{51}$$

The swimming speed is found to be $U_s = -\frac{1}{4}\varepsilon^2$, in agreement with the large R limit of Tuck.

Analysis of II

For the stretching planar sheet we show now that the problem reduces to an expansion in powers of ε and R^{-1}, so that the limit of $\overline{\Psi}$ for large R exists as an expansion in ε. Therefore for problem II we can compute the inviscid limit of the flow.

To see this we need only note that boundary conditions are invariant under the passage to inner variables:

$$\overline{\Psi}(\xi + \varepsilon\cos\xi, 0) = 0, \overline{D}\overline{\Psi}(\xi + \varepsilon\cos\xi, 0) = \varepsilon\sin\xi. \tag{52}$$

Thus we may define inner and outer expansions by

$$\overline{\Psi} \sim \varepsilon\overline{\Psi}_1 + \varepsilon^2\overline{\Psi}_2 + \ldots \text{ (inner)}, \tag{53}$$

$$\psi \sim R^{-1/2}\left[\varepsilon\psi_1 + \varepsilon^2\Psi_2 + \ldots\right] \text{ (outer)}. \tag{54}$$

It is easy to see that

$$\overline{\Psi}_1 = \frac{1}{\sqrt{2}}\left[\cos\xi + \sin\xi - e_{-Y}(\cos X - \sin X)\right], \tag{55}$$

$$\psi_1 = \frac{1}{\sqrt{2}}(\cos\xi + \sin\xi)e^{-\eta}. \tag{56}$$

Also, since $\overline{D}\overline{\Psi}_2(\xi,0) = -\cos\xi\,\overline{D}\partial_\xi\overline{D}_1(\xi,0) = \cos^2\xi$, we have

$$\langle\overline{D}^3\overline{\Psi}_2\rangle = -\langle\partial_\xi\overline{\Psi}_1\,\overline{D}^2\overline{\Psi}_1\rangle. \tag{57}$$

Sustituting and integrating once,

$$\langle\overline{D}^2\overline{\Psi}_2\rangle = -\frac{1}{2\sqrt{2}}\left[e^{-2Y} - e^{-Y}(\sin Y - \cos Y)\right]. \tag{58}$$

We therefore obtain

$$\overline{D}\overline{\Psi}_2(\infty) + \frac{1}{2} = -\frac{1}{4}, \tag{59}$$

yielding a swimming velocity of $U_s = \frac{3}{4}\varepsilon^2$. The result of [2] for problem II at finite R is

$$U_s = \frac{1}{2}\varepsilon^2\Big[\frac{3F(R)-1}{2F(r)}\Big]. \tag{60}$$

which agrees in the limit of large R.

Higher-order terms for problem I

Although the expansions in I require that $\varepsilon R^{1/2} \ll 1$, we are entitled to make this parameter far larger than $R^{-1/2}$. Since terms with subscripts $n0$ vanish if $n > 1$, the terms with subscripts $11, 21, 31, \ldots$ can be made to dominate. We may think of these as expansions of $\overline{D\Psi}$ for fixed but small δ of a function of order $R^{-1/2}$, with a formal error of order R^{-1} at large R. In particular at $\overline{\eta} = \infty$ we should have

$$\langle\overline{D\Psi}\rangle \equiv U(\delta) \sim U_1(\delta)R^{-1/2} + U_2(\delta)R^{-1} + \ldots. \tag{61}$$

Now we know that for small δ, we have

$$U_1 = O(\delta^3),\ \ U_2 = O(\delta^2) \tag{62}$$

We claim that U_1 must in fact vanish identically and that U_2 contains only even powers of δ. Indeed in the primitive dimensionless form of problem I, the powers of trigonometric functions are introduced through the nonlinearity of the momentum equation and through the displacement $\varepsilon\sin\xi$ of the boundary. The powers add under nonlinear interaction. In inner variables the boundary terms increase the powers of trig functions and of δ in concert. Consequently $\overline{\Psi}_{mn}$ will contain even powers of trig functions only if m is even.

To show that $U_1 = 0$ we consider the series in δ of the solution of the following linear problem:

$$\overline{D}^2\big[\partial_\xi\overline{\Psi} + \overline{D}^2\overline{\Psi} + \delta\,\cos\xi\,\overline{D\Psi}\big] = 0, \tag{63}$$

$$\partial_\xi\overline{\Psi}(\xi,\overline{\eta})\big|_{\overline{\eta}=\delta\sin\xi} = \delta\cos\xi,\ \overline{D\Psi}(\xi,\delta\sin\xi) = 0. \tag{64}$$

The function $\overline{\Psi}$ grows like $\overline{\eta}$ at ∞, since higher powers correspond to $\overline{\Psi}_{mn}$ with $n > 1$. We may therefore average (63) with respect to ξ, integrate by parts with respect to ξ, and integrate from $\overline{\eta} = \infty$ twice to obtain

$$\overline{D}^2\langle\Psi\rangle = \delta\langle\sin\xi\,\partial_\xi\overline{D\Psi}\rangle. \tag{65}$$

We assume a series solution

$$\overline{\Psi} \sim \sum_{n=1}^{\infty}\delta^n\overline{\Psi}_n,\ \overline{\eta} \geq 0. \tag{66}$$

where $\overline{\Psi}_n = \overline{\Psi}_{n1}$ in the earlier notation; we have already computed the first three terms of the series. We write the second of (64) as

$$\overline{D\Psi}(\xi,0) = -\sum_{n=1}^{\infty}\frac{1}{n!}\sin^n\xi\overline{D}^{n+1}\overline{\Psi}(\xi,0). \tag{67}$$

Integrating (65) from 0 to ∞ after making use of (63) we have

$$U_1(\delta) - \overline{D\Psi}(\xi,0) = -\delta\langle\sin\xi\partial_\xi\overline{\Psi}(\xi,0)\rangle$$

$$= \delta\langle\sin\xi\overline{D}^2\overline{\Psi}(\xi,0)\rangle + \delta^2\langle\sin\xi\cos\xi\overline{D\Psi}(\xi,0)\rangle$$

$$= \delta\langle\sin\xi\overline{D}^2\overline{\Psi}(\xi,0)\rangle - \frac{1}{2}\langle\sin^2\xi\overline{D}^3\overline{\Psi}(\xi,0)\rangle$$

$$+\frac{1}{2}\delta^3\langle\sin^2\xi\cos\xi\overline{D}^2\overline{\Psi}\rangle. \tag{68}$$

Continuing in this way, we obtain

$$U_1(\delta) = \langle\overline{D\Psi}(\xi,0)\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}\delta^n\langle\sin^n\xi\overline{D}^{n+1}\overline{\Psi}(\xi,0)\rangle = 0, \tag{69}$$

since this right-hand side is the average of the null condition in (67).

It follows that we have the following estimate of the error in the swimming speed for problem I:

$$U_s \sim -\frac{1}{4}\delta^2R^{-1} + O(\delta^4R^{-1}) + O(\delta^2R^{-3/2}), \tag{70}$$

or in terms of ε and R,

$$U_s \sim -\frac{1}{4}\varepsilon^2 + O(\varepsilon^4R) + O(\varepsilon^2R^{-1/2}). \tag{71}$$

The second error term comes from the expansion of Tuck's result for large R.

The inviscid problem

We have seen that problems I and II differ fundamentally in the nature of the inviscid limit. This limit exists and is computable from matched expansions in the case of II, but for problem I this limit is not accessible by matched expansions. the reason lies of course in the separation phenomenon, which must occur at large enough R regardless of the size of ε. If an inviscid fluid is assumed, a potential flow problem may be solved for I as an expansion in ε. It is easily shown that the expansion of the velocity potential in this case, satisfying the tangency condition at the boundary,

$$\frac{\partial\phi}{\partial\eta} = \varepsilon\Big(-1+\frac{\partial\phi}{\partial\xi}\Big)\cos\xi,\ \ \eta = \varepsilon\sin\xi. \tag{72}$$

is

$$\phi = \varepsilon e^{-\eta}\cos\xi + \frac{1}{2}\varepsilon^2 e^{-2\eta}\ \sin 2\xi + \dots. \tag{73}$$

Here ϕ is in units ω/k^2. This corresponds in dimensional units to a kinetic energy $\rho(\omega b)^2/(4k)$ per unit length of sheet. D'Alembert's paradox of zero pressure drag eliminates the possibility of swimming in an inviscid fluid. For problem II, moreover, the fluid is not disturbed at all by the stretching sheet. Since the inviscid limit exists in that case, we have an excellent example of the inviscid limit of a flow being distinct from the corresponding flow of an inviscid fluid.

Recoil locomotion of a sheet

Saffman [4] has discussed swimming of a deformable finite body in an inviscid fluid. This can occur from the relative movements of the center of volume of a body, and its center of mass (including the virtual mass of the fluid). We discuss now a related problem of a deforming sheet in two dimensions. Consider two massless sheets, mirror symmetric with respect to the x-axis, the upper sheet having the equation $y = 2b + b\sin(\omega t+\theta)\sin\big(k(x-\mathcal{X}(t))\big) \equiv Y(x,t)$, where we include an arbitrary phase θ. Fluid occupies the domain $|y| > Y(x,t)$. On the line $y = 0$ we place small bodies of mass m at positions $x = x_n(t) = 2\pi n/k + \mathcal{X}(t) = L\sin 2\omega t, n = 0, \pm1, \pm2, \dots$. The machinery moving the masses is assumed to be attached to the sheets. As the masses move back and forth, the sheets move by recoil in the presence of the varying virtual mass of the fluid, and swimming can result. The function $\mathcal{X}(t)$ tracks the x-coordinate of a point on a sheet.

To compute the swimming speed we assume that the system is started from rest, so that total momentum of fluid and masses per unit length of sheet must remain zero. Assuming again $bk = \varepsilon \ll 1$, the dimensional potential to first order is $-\dot{\mathcal{X}}b\sin(\omega t+\theta)e^{-ky}\cos(x-\mathcal{X})$. This yields the kinetic energy $\pi\rho\dot{\mathcal{X}}^2 b^2\sin^2(\omega t+\theta)$ over one wavelength of sheet, allowing for the fluid on both sides of the body. The virtual mass of the fluid is therefore

$$M(t) = 2\pi\rho b^2\sin^2(\omega t+\theta). \tag{74}$$

Conservation of momentum then implies

$$m(\dot{\mathcal{X}} + 2\omega L\cos 2\omega t) + \dot{\mathcal{X}}M(t) = 0, \tag{75}$$

or

$$\dot{\mathcal{X}} = \frac{2\omega m L\cos 2\omega t}{m+\pi\rho b^2[1-\cos 2(\omega t+\theta)]}. \tag{76}$$

The dimensionless form of this expression is

$$\dot{\mathcal{X}}^*_\tau = \frac{-2L^*\cos 2(\tau-\theta)}{1+\frac{\pi\varepsilon^2}{m^*}(1-\cos 2\tau)}, \tag{77}$$

where $\tau = \omega t, \mathcal{X}^*_\tau = \dot{\mathcal{X}}k/\omega, L^* = kL$ and $m^* = mk^2/\rho$. Assuming L^*, m^* are order unity the leading term of the expansion in ε of the swimming velocity is, averaging over time,

$$\frac{\omega}{k}U_s = -\pi\frac{L^*}{m^*}\cos 2\theta\varepsilon^2. \tag{78}$$

We thus get a swimming velocity through recoil which is comparable to that achieved in Stokes flow, at least for the order of the parameters we have assumed.

Recoil swimming in a viscous fluid

In a ongoing collaboration with Tadashi Tokieda we are addressing the following question: what happens when the inviscid model of this section is placed in a viscous fluid? Diffusion of momentum to infinity is then possible, but the expansion methods of the present paper can be used. Tentative results indicate that, in the case $L^* = O(\varepsilon)$, a swimming speed of order $\varepsilon^3\sqrt{R}$ results, indicating that viscosity actually increases the velocity above the inviscid case. This result, along with an application of matched asymptotic expansions to recoil swimming in two dimensions, will be reported elsewhere.

Discussion

The purpose of this investigation has been to clarify the nature of the of the high Reynolds number limit of the problem of the swimming sheet as formulated by G.I. Taylor [1]. In this limit the problem is characterized by the formation of boundary layers, and in the case of a propagating wave of amplitude b (problem I), straightforward analysis of the problem is possible, but only if the boundary layer thickness is large compared to the wave amplitude. This dictates our expansion in small δ in place of the small ε expansion of Tuck [2]. For a wave of stretching on a planar sheet (problem II), no such assumption is necessary and a boundary-layer limit exists. This represents one of the few problems in locomotion which can be essentially completely solved at any Reynolds number.

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