On vortex stretching and the global regularity of Euler flows I. Axisymmetric flow without swirl

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March 6, 2006

Abstract

The question of vortex growth in Euler flows leads naturally to the emergence of paired vortex structure and the "geometric" stretching of vortex lines. In the present paper, the first of two papers devoted to this question, we examine bounds on the growth of vorticity in axisymmetric flow without swirl. We show that the known bound on vorticity in this case, exponential in time, can be improved for large time by adhering closely to the geometric constraints imposed by the symmetry of the flow. and the conservation of the support of vorticity. Under appropriate conditions, the vorticity is shown to grow no faster that $O(t^2)$. The kinematic vortex structure used to obtain this bound does not, however, conserve kinetic energy. If energy conservation is imposed, but not that of support volume, the bound is reduced to $O(t^{4/3})$. It appears that optimizing vorticity conserving both energy and volume will involve filamentary structures.

We further propose that in the absence of the symmetry of the present class of flows, conservation of energy should be dropped from the local analysis of stretching of paired structures having variable stretching rates, and replaced by conservation of total energy, an idea which is explored further in the second paper.

1 Introduction and motivation

The question of global regularity of three-dimensional solutions of the incompressible Euler equations continues to be of considerable interest to both mathematicians and fluid dynamicists. A recent assessment of the problem from the analytical viewpoint may be found in [1], see also [2] aand [8]. Numerical studies have been difficult, occasionally suggestive of finite time singularities, but inconclusive. Analytical studies have been highlighted by the key condition of Beale, Majda, and Kato [3], who showed that if a finite time singularity occurs, the integral of the maximum modulus of vorticity up to the singularity time must diverge. (For the proof, other similar conditions, and critical comment see [1].) Constantin, Fefferman, and Majda [2] subsequently extended the conditions to the geometry of the vorticity field, and specifically to the direction field of unit tangent vectors to vortex lines. Recently Deng *et al.* [5] have used similar ideas to argue non-existence of singularities in some of the numerical experiments.

The present paper grew from a study of [1]–[5], and is motivated by several "working hypotheses" concerning Euler flows. First, the lack of convincing numerical experiments, as well as physical intuition, suggest that finite time Euler singularities are rare events in the context of the initial-value problem for Euler's equations.

Working hypothesis 1 *Almost all Euler flows are free of finite time singularities.*

That is, if a suitable measurable space of smooth initial conditions is given, those initial conditions leading to singularities should constitute a set of measure zero.[1] Should a singular solution exist, if this hypothesis were true, it would be unimportant physically in that solutions with nearby initial conditions would in general be free of singularities.

The intuitive reason for this view lies in the non-local nature of the mutual stretching of vortex lines which seems to be needed in order promote a finite time blowup. Let vortex element A act on vortex element B so that lines of B are stretched at a rate proportional to the vorticity at A. The idea is then for B to do the same to A, so that the time rate of change of vorticity in either element is proportional to the vorticity squared, leading to blowup of vorticity like $1/(t_* - t)$. We suggest that such a construct, if indeed obtainable, would be highly unstable to the slightest perturbation of the vortex lines and is likely to represent a negligible set of Euler flows as suggested above.

However, even rare Euler singularities are worthy of study, and in this paper we try to understand how vorticity could self stretch optimally in a manner compatible with the basic fluid dynamics. We postulate that the vortex line topology needed to attack this problem need not be too complicated:

Working hypothesis 2 *The maximal growth rate of vorticity in almost all Euler flows can be estimated from flows whose vortex lines have a relatively simple topology, for example, from flows all of whose vortex lines are simple closed, unlinked loops (unknots).*

In [4] it is shown that the direction field of vorticity cannot be too regular if a finite time singularity occurs. Here we shift the focus, and instead attempt to estimate the maximum growth achieved in flows of simple topology. We say a solenoidal field $\omega_0(\mathbf{x})$ is *flow-equivalent* to a solenoidal field $\omega_T(\mathbf{x})$ if there a positive number T and a solenoidal smooth solenoidal field $\mathbf{u}(\mathbf{x},t), 0 \leq t \leq T$ such that the solution $\omega(\mathbf{x},t)$ of

$$\omega_t + \mathbf{u} \cdot \nabla\omega - \omega \cdot \nabla\mathbf{u} = 0, \nabla \cdot \omega = 0,\ 0 \leq t \leq T \qquad (1)$$

has the property that $\omega(\mathbf{x},0) = \omega_0(\mathbf{x}), \omega(\mathbf{x},T) = \omega_T(\mathbf{x})$. That is, the vorticity fields ω_0, ω_T are flow equivalent if ω_T is reached from ω_0 by carrying ω as a

[1] A simple example of a such a measurable space would be spatially periodic flows with sufficiently rapid convergence of the Fourier sums.

'frozen in" vector field under the action of the flow $\mathbf{u}$. The Lagrangian map determined by $\mathbf{u}, 0 \leq t \leq T$, establishes ω_T as the image of ω_0 under a diffeomorphism. The velocity fields corresponding to these two vorticity fields are said to be *isovortical* [6].

The class of initial vorticity fields we propose to explore are those which are flow equivalent to axisymmetric flow without swirl. Axisymmetric flow *with* swirl has often been put forward as candidate for singularities, as has the related problem of 2D, stratified, incompressible flow under the Boussinesq approximation , see [1].

Velocity fields which are isovortical to axisymmetric flow without swirl can have enormous complexity, yet they have the simple topology of our second hypothesis– every vortex line is a closed loop linking with no other vortex loop. But it is fair to ask why a simple topology is of any use if the velocity field can be so complex. In the present paper we shall utilize the topology explicitly in the rearrangement of vorticity, in the quest for maximal vortex stretching. Rearrangments of vorticity can be attempted under varying constraints, kinematic, dynamical, or energetic, without attempting to solve Euler's equations exactly, and this flexibility can be exploited most directly if topological constraints are eliminated from the outset.

We shall in the present paper focus on the simplest of these flows, namely axisymmetric flow without swirl itself. Any axisymmetric flow having no swirl is known to exist globally in time, and a very direct proof of this fact is given in [1]. We deal here only with flows in R^3, and give the proof in detail, since it is a principal motivator for our work.

The proof utilizes, in a way which will be clear below, two essential facts, the first for Euler flows in general, the second for axisymmetric flow without swirl in particular: (1) Since vorticity is a "frozen in" vector field, the volume of its support is conserved in time. (2) $r^{-1}\omega_\theta(\mathbf{x}, t)$ is a material invariant of the flow, where $r = (x^2 + y^2)^{1/2}$ is the cylindrical polar radius. Thus, the vorticity associated with with any vortex line (ring) at time t, can be directly expressed in terms of the current radius of the ring, its initial radius, and the initial ω_θ.

In these axisymmetric flows without swirl flow the vorticity is $(0, 0, \omega_\theta)$ in cylindrical polar (z, r, θ) coordinates, and the velocity has the form $(u_z, u_r, 0)$. Let the initial vortical field $\omega_{\theta 0}(z, r)$ be smooth, bounded, and and supported on a region of volume finite V_0. It follows that the support of the vorticity at any future time has volume V_0. We further assume $|r^{-1}\omega_{\theta 0}(z, r)| < C$ on its initial support.

We can then estimate $\max(|\mathbf{u}|)$ over all space as follows: using the Biot-Savart representation of the velocity in terms of vorticity,

$$\max(|\mathbf{u}|) \leq \Big|\frac{1}{4\pi}\int_{|\mathbf{y}|\leq R_0} \frac{\mathbf{y}\times\omega'}{y^3} dV'\Big| + \Big|\frac{1}{4\pi}\int_{|\mathbf{y}|\geq R_0} \frac{\mathbf{y}\times\omega'}{y^3} dV'\Big| \quad , \mathbf{y} = \mathbf{x} - \mathbf{x}'. \tag{2}$$

Clearly

$$\max(|\mathbf{u}|) \leq \max_{supp}(|\omega_\theta(z, r, t)|)[4\pi R_0 + V_0 R_0^{-2}]. \tag{3}$$

If we set $R_0^3 = V_0$, we get $\max(|\mathbf{u}|) \leq c_1 \max_{supp} |\omega|$, where $c_1 = (1 + 4\pi)R_0$. Now in this Euler flow

$$\omega_\theta(z, r, t)/r = \omega_\theta(r_0, z_0, 0)/r_0, \tag{4}$$

where (z, r) and is the terminal point of a fluid particle which started at (z_0, r_0). Now let $R(t)$ be the radius of the support at time t. Then we have

$$dR/dt \leq \max(\mathbf{u}) \leq c_1 \max_{supp}(\omega_\theta) \leq Cc_1 R. \tag{5}$$

By Grönwall's lemma, the radius of the support, hence the maximum vorticity, grows at most exponentially in time.

The proof thus utilizes the conservation of the volume of the support to extract a bound of velocity in terms of the global maximum of $|\omega_\theta|$ at a fixed time, thereby obtaining a bound on the maximal rate of expansion of the support; then the maximum velocity is expressed in terms of the radius of the support using the material property of $r^{-1}\omega_\theta$. The latter property is used only at the end, not in estimating $\max(\mathbf{u})$ in terms of the vorticity. The proof thus does not account for a simple fact about the ultimate fate of an expanding vortex ring: the only way for the ring to keep expanding is for there to be nearby vorticity, which can induce the necessary advecting velocity field. However, as the ring is expanding under the constraint of conservation of volume, necessarily the inducing velocity would have the greatest effect if it were confined to a toroidal neighborhood of the expanding ring. This leads to a distinctly different estimate of the ultimate growth rate of vorticity in axisymmetric flow without swirl, as we shall show below. The exponential estimate can be improved by more detailed tracking of the material invariant. In seeking to lower the bound on the growth rate, our interest is in the symmetric flow only as a test case for the reduction of growth under the addition of constraints.

2 Axisymmetric flow without swirl

Let the initial vorticity have an initial support of volume V_0, i.e. the points where vorticity is non-zero constitute a volume V_0. Suppose that $-c_1 \leq \omega_\theta(\mathbf{x}, 0) \leq c_2$ for some positive constants c_1, c_2, and let the region of the support where $\omega_\theta \geq 0$ have volume V_{0+}, that where $\omega_\theta < 0$ have volume $V_{0-} = V_0 - V_{0+}$. We suppose that $r^{-1}|\omega_\theta(\mathbf{x}, 0)| \leq C$.

2.1 Construction of the cocoon with conservation of support volume

Consider any vortex ring at time t. Taking the z axis as the axis of symmetry, we may assume the ring has radius r at time t, and lie on the plane $z = 0$. We refer to this ring as the *core ring*. Let $V/2 = \max(V_{+0}, V_{0-})$. It is clear that to maximize the rate of growth at time t of the ring in question, we can take rings of negative vorticity $\omega_\theta = -Cr$ distributed over a volume $V/2$ in $z \geq 0$, and rings of positive vorticity $\omega_\theta = +Cr$ distributed over a volume $V/2$ in $z \leq 0$. Note that θ increases counterclockwise looking onto the x, y plane from $z > 0$, so by the right-hand-rule a negative ω_θ in $z > 0$ induces a positive u_r (and a negative u_z) at the core ring.

Consider now the value of u_r induced at the core ring by a ring of radius ρ and cross-sectional area $2\pi\rho dA$ carrying vorticity $-C\rho$ at height $z = \zeta > 0$. From the Biot-Savart law one finds

$$u_r(r,0,t) \leq \frac{C\rho^2|\zeta|}{4\pi}\Big[\int_{-\pi}^{+\pi}((r-\rho)^2 + 2r\rho(1-\cos\psi) + \zeta^2)^{-3/2}d\psi\Big]dA \tag{6}$$

Since $1-\cos\psi \geq k^2\psi^2, ; |\psi| \leq \pi,\ k = \sqrt{2}/\pi$, we may make this substitution and carry out the integral with the range extended from $[-\pi, +\pi]$ to $[-\infty, +\infty]$, to obtain

$$u_r(r,0,t) \leq \frac{C|\zeta|\rho^{3/2}}{4\sqrt{r}}((r-\rho)^2 + \zeta^2)^{-1}dA \tag{7}$$

We now want to optimize an arrangement of rings about the core ring which, by carrying the maximal vorticity of each sign in the appropriate half plane, will clearly be causing the maximal possible stretching of the core ring, subject only to the constraint on the volume of the support. The optimal configuration will be termed the *cocoon* of the core ring.[2] In order to make the variational problem the most transparent possible, we make a few technical simplifications.

We introduce local polar coordinates in the r, z plane, defined by $\rho - r = R\cos\Theta, \zeta = R\sin\Theta$. Then, since

$$\begin{aligned} u_r &\leq \frac{C|\sin\Theta|(r+R\cos\Theta)^{3/2}dRd\Theta}{4\sqrt{r}} \\ &\leq \frac{C}{4}|\sin\Theta|(r+R\cos\Theta)(1+R/r)^{1/2}dRd\Theta, \end{aligned} \tag{8}$$

we seek to maximize

$$U = \int_{\mathcal{A}} f(R,\Theta)dRd\Theta,\ f = \frac{C}{4}|\sin\Theta|(r+R\cos\Theta)(1+R/r)^{1/2}, \tag{9}$$

subject to the volume constraint

$$V = \int_{\mathcal{A}} g(R,\Theta)dRd\Theta,\ g = 2\pi(r+R\cos\Theta)R. \tag{10}$$

Here $\mathcal{A}$ is a set to be determined. We may assume by symmetry that $\mathcal{A}$ is mirror symmetric in the plane $z = 0$, since the vorticity field of the cocoon is odd in z. Also, we may assume that the core ring radius is as large as we like when we begin the tracking of the cocoon, since otherwise vorticity is bounded by a fixed constant for all time. In addition, we need the following preliminary result:

[2] I thank Peter Constantin for suggesting this descriptive term. I am also indebted to the class of the Spring 2004 Fluid Dynamics class at the Courant Institute, who patiently endured an unexpected foray into topological fluid dynamics, where some of the ideas presented here were developed. The lecture notes are available at http://www.math.nyu.edu/faculty/childres/fluids22004.html.

Lemma 1 *We may assume that the half $\mathcal{A}-$ of $\mathcal{A}$ in $z \geq 0$ is a region of the form $0 \leq R \leq \mathcal{R}(\Theta), 0 \leq \Theta \leq \pi$. That is, the region can be assumed to be starlike with respect to the core ring.*

To show this, suppose that Θ is fixed and note that the intersection of $\mathcal{A}$ with the ray determined by Θ determines a set function $\phi(R)$ equal to 1 in $\mathcal{A}$ and otherwise 0. Consider then two choices of ϕ, either $\phi_1 : 0 \leq R \leq \mathcal{R}$, or else a set of disjoint intervals ϕ_2, such that

$$\int_0^\infty \phi_2 g dR = \int_0^\infty \phi_1 g dR. \tag{11}$$

We then want to show that

$$\int_0^\infty \phi_2 f dR > \int_0^\infty \phi_1 f dR. \tag{12}$$

But this follows immediately from the fact that f, g are positive functions on the support of $\phi_{1,2}$ and that and f/g is a positive multiple of a decreasing function of R, namely $(1/R^2 + 1/(Rr))^{1/2}$.

Using the lemma, and the mirro symmetry of the cocoon, we may formulate the optimization problem as the variational problem for the boundary $\mathcal{R}(\Theta), 0 \leq \Theta \leq \pi$, given by

$$\delta \int_0^\pi \int_0^\mathcal{R} \big(f(R,\Theta) - \lambda g(R,\Theta)\big) dR\Theta, \tag{13}$$

with scalar multiplier λ.

The extremal of this variational problem, $\mathcal{R}(\theta$, satisfies

$$(r + \mathcal{R}\cos\Theta)\big(K \sin\Theta \sqrt{1 + \mathcal{R}/r} - \mathcal{R}/r\big) = 0, \tag{14}$$

where $K = \frac{C}{8\pi\lambda r}$. If r is sufficiently large, $r + \mathcal{R}\cos\Theta)$ stays nonnegative and the unique extremal is

$$\mathcal{R}(\Theta) = r\sqrt{K^2 \sin^2\Theta + \frac{K^4 \sin^4\Theta}{4}} + \frac{K^2 \sin^2\Theta}{2}, \tag{15}$$

The variational equation $\mathcal{R}^2 = r^2 K^2 \sin^2\Theta(1 + \mathcal{R}/r)$ yields the volume constraint which determines K:

$$V = 2\pi r^3 \int_0^\pi K^2 \sin^2\Theta(1 + \mathcal{R}(\Theta)/r) d\Theta. \tag{16}$$

Now in view of (15) we see $\min_{0<\Theta<\pi} r + \mathcal{R}\cos\Theta$ ceases to be positive when

Let us introduce a length L such that $V = 2\pi L^3$. Then the integral (16) defines a function $K(r*)$, where $r* = r/L$. From (15) and the calcul.ated values of $K(r*)$ we find that $\min_{0<\Theta<\pi} r + \mathcal{R}\cos\Theta$ ceases to be positive when

$r* < .5177$ approximately We thus obtain, taking into account both mirror-symmetric halves of the cocoon, for $r* > .5177$, the differential inequality

$$\frac{dr*}{dt} \leq \sup U \leq \frac{CLr*^2}{3} \int_0^\pi \sin\Theta\big[\big(1+\frac{\mathcal{R}}{r}\big)^{3/2} - 1\big] d\Theta \equiv \frac{CLr*^2}{3} \mathcal{U}(r*), \quad (17)$$

where we define the function $\mathcal{U}(r*)$. We show this relation in figure 1, along with the cocoons at various values of r/L.

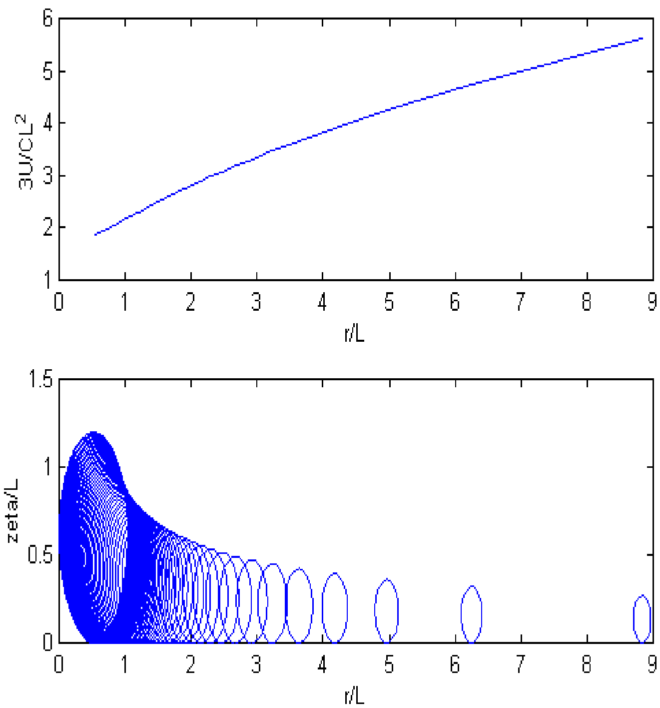

Figure 1. Top: $\frac{3}{CL^2}\frac{dr}{dt}$ (as defined by (17)) versus r/L. Bottom: Cocoon shape for various position of the core ring. The cocoon is mirror symmetric with respect to the r/L line.

From the behavior for large r/L (or small K), we obtain from (16) $K \sim \frac{\sqrt{V}}{\pi r^{3/2}}$, and from (17) $dr/dt \leq \frac{CK\pi r^2}{4}$, yielding the estimate

$$\frac{dr}{dt} \leq \frac{C}{4}\sqrt{Vr}, \ r \to \infty. \quad (18)$$

Thus $d\sqrt{r}/dt \sim\leq \frac{C}{8}\sqrt{V}$ for large r. With $|\omega_\theta(r, z, t)| \leq Cr$ we obtain the following result:

Theorem 1 *For axisymmetric flow with initial support volume V and initial vorticity satisfying $|\omega_\theta/r| \leq C$, there is a constant C_1 depending only upon V, C such that*

$$\sup|\omega_\theta| \leq C(\frac{C}{8}\sqrt{V}t + C_1)^2. \tag{19}$$

Thus vorticity grows no faster than $O(t^2)$ for large time.

To establish the theorem, we may assume that at time $t = 0$ the core ring is at a position such that the cocoon satisfies $\min_{0\leq\Theta\leq\pi}[r + \mathcal{R}\cos\Theta] \geq 0$. Thus dr/dt is bounded by the curve shown at the top of figure 1, with the asymptotic behavior given by (18), and the theorem follows.

2.2 Remarks

We note first that the factor $C/8$ in (19) maybe replaced by $C/(4\pi)$. This is because if only the case $r \gg |r - \rho|$ is considered for (6), the factor $2(1 - \cos\psi)$ in the integrad may be rplaced by ψ^2 and the integration extended to $-\infty, +\infty$, effectively inserting a factor $2/\pi$.

While the construction of the cocoon is based upon geometric constraints associated with Euler flows, it is a local construction (in time) which has no direct relation to the evolution of the flow. Thus, for example, the core ring is here a "test ring" whose expansion rate in r is maximized. In the construction, cocoon vorticity is in fact placed at larger values of r. In practice the most rapidly growing ring would leave vorticity behind, and there would be a arrangement of rings which expanding at a rate well below our upper bound.

This can be illustrated by adapting a well-known example of a propagating vortex dipole, namely the 2D vortex structure described by Batchelor [7]. The vorticity is contained within the circle $r = a$, and is given by $\omega = -Ak^2 J_1(k\rho)\sin\theta$, where A is an arbitrary constant, and $J_1(ak) = 0$. Here (ρ, θ) are local polar coordinates. We take the smallest ak satisfying the last condition, namely $ak = 3.83$ approximately, to obtain one sign of vorticity in each half-plane. On $r = a$ the velocity is the same as for irrotational flow past a circular cylinder, provided that the cylinder moves with speed $U = -\frac{1}{2}AkJ_0(ka)$.

We now take this flow as that of any cross section of a slender toroidal ring, see figure 2. As the ring expands, a must diminish to conserve volume, but we may consider this as a a more closer than our cocoon to realizing an Euler flow. Now $2\pi r \cdot \pi a^2 = V$

$$C = \frac{Ak^2}{r} \max_{0\leq\rho\leq a} |J_1(k\rho)|. \tag{20}$$

From these relations and the properties of the Bessel functions J_0, J_1 we obtain

$$U \approx .02C\sqrt{(Vr)}, \tag{21}$$

the factor .02 is indeed well below the the $1/(2\pi)$ in our bound.

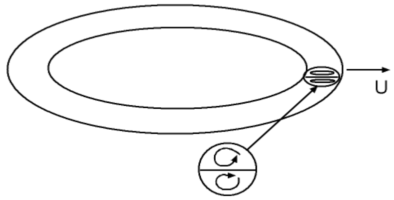

Figure 3. Expanding vortex structure yields the t^2 behavior.

The cocoon based upon conservation of support of vorticity (as well as the toroidal ring construction just described) is deficient in another important aspect, namely *it does not conserve the kinetic energy of the flow.* Thus it cannot be sharp for Euler flows.

To see this, recall that the kinetic energy of an axisymmetric vortical field in R^3 in a flow without swirl can be expressed in terms of vorticity in the form

$$E_0 = \frac{\rho}{8\pi}\int_V\int_{V'}|\mathbf{r}-\mathbf{r}'|^{-1}\omega_\theta\omega_\theta'\mathbf{i}_\theta\cdot\mathbf{i}_{\theta'}dVdV'. \tag{22}$$

This can be expressed in cylindrical polar coordinates as

$$E_0 = \frac{-\rho}{8\pi}\int_V\int_{V'}\omega_\theta\omega_\theta'\cos\psi|(r+r'))^2-4rr'\sin^2(\psi/2)+(z-z')^2|^{-1/2}r'dr'd\theta'dz'rdrd\theta dz, \tag{23}$$

where $\psi=\theta-\theta'$. When $r,r'\gg|r-r'|$ the integral with respect to θ' may be evaluated approximately as a complete elliptic integral, yielding

$$E_0 \approx \frac{\rho}{2}\int_V\int_{V'}r\omega_\theta\omega_\theta'\log\frac{64r^2}{(r-r')^2+(z-z')^2}dr'dz'drdz. \tag{24}$$

We now study a configuration for our cocoon, where vorticity is $-Cr$ in the upper half-plane, and $+Cr$ negative in the lower. Then

$$E_0 \approx \frac{\rho C^2r^3}{2}\int_{A_0}\int_{A_0'}sgn(zz')\log\frac{64r^2}{(r-r')^2+(z-z')^2}drdzdr'dz'. \tag{25}$$

Assuming now that the support of vorticity is an even function of z, wee see that the contribution $\log 64r^2$ from the integrand will not contribute. As $r\to\infty$, the linear dimension of the cocoon cross section shrinks by the factor $r^{-1/2}$, so we see that E_0 *grows linearly in* r.

It is natural then, to seek to improve (19) by adding the constraint of conservation of energy to the cocoon construction. We shall argue below that for the construction used above, where vorticity is replaced by its upper bound, and the cocoon has piecewise constant ω_θ/r, that this leads to degenerate cocoons with infinitesimal concentrations of vorticity which carry no energy. We shall refer to these concentrations as *filaments* This suggests that, at least in axisymmetric flow without swirl which in fact consists of domains where ω_θ/r is piecewise constant, the largest vorticity for large time is found in regular structures which conserve energy but not the support of vorticity. We will eventually be guided by this result in addressing all Euler flows isovortical to axisymmetric flow without swirl, and so introduce

Working hypothesis 3 *An improved bound of vorticity, relative to that for the cocoon of invariant support, is obtained by the cocoon of invariant kinetic* *energy. This cocoon may be extended so as to also conserve the support of vorticity, either by the addition of filamentary vorticity, or else by extending the admissible vortical fields. In the case of axisymmetric flow without swirl, this would be accomplished by allowing vortical distributions with non-constant ω_θ/r.*

The remainder of this paper is devoted to investigating these issue. We shall not attempt the same level of rigor as we sought in the construction of the cocoon conserving support. We may assume that r becomes as large as we want and therefore we may restrict attention to $r \gg L$ where the cocoon construction involves a thin toroidal structure. For axisymmetric flow without swirl and cocoons of piecewise constant ω_θ/r, we will first determine the regular cocoon conserving energy, then indicate the filamented extension which conserves support as well. Finally we shall argue for the validity of this extremal from upon a model problem based upon a thin-sheet approximation.

2.3 The cocoon which conserves energy

While the support of vorticity is independent of the magnitude of the vorticity on each ring, the energy is not, and we first argue that the cocoon may again be constructed by considering a structure with vorticity $\pm Cr$. Let us first suppose that a cocoon has been found which maximizes U for a fixed energy initial E_0, with $|\omega_\theta| \leq rC$. This extremal maximizes $Ur^{3/2}/\sqrt{E/\rho}$. If, at the optimum, vortex rings in $z > 0$ carry vorticity $-Cr$ and those in $z < 0$ have $-Cr$, then we call $E_0 = E_c$ the *cocoon energy*. Otherwise, the value of U so obtained will be smaller than that obtained by assigning vorticity $-Cr$ to every ring above the core ring, and $+Cr$ to every ring below the core ring. This new structure will have a larger energy than E_0, since the previous distribution was optimal, and this now defines the cocoon energy E_c. The cocoon energy will be conserved in the dependence of the cocoon upon r. This is because once $r \gg V^{1/3}$ the cocoon is defined locally and shrinks through self-similar structure, being simply scaled down by the linear factor $r^{-3/4}$ as r increases.

Now this new U is bounded above by that value obtained by maximizing U subject to $E = E_c$ and vorticity $\pm Cr$. That is, this last optimization replaces the boundary of the first extremal by a new one. For this latter construction we are essentially returned to the construction with fixed support, only that now energy of the system replaces volume as the conserved quantity. The energy involved is now cocoon energy, which is larger in general than physical energy.

We now claim that a the result analogous to lemma 1, allowing the admissible cocoons of the form $R = \mathcal{R}(\theta)$, holds under energy conservation. The proof compares small vorticity elements in the local cocoon cross section. Let a small element dA of the cross section be located at (R_1, θ), and a second element, mirror symmetric with respect to $z = 0$ with the first be selected at $(R_1, -\theta)$. Now let these elements be moved to (R_2, θ) and $(R_2, -\theta)$ respectively, where $R_2 < R_1$. The positive"self-energy" of the two elements is unchanged by this shift, but the "interaction energy", which is here negative owing to the signs of the vorticity, is enhance, i.e. becomes more negative, since $\log R_2^{-1} > \log R_1^{-1}$.

Consequently, to maximize $\frac{Ur^{3/2}}{\sqrt{E/\rho}}$ subject to $E = E_c$ we may assume the geometry of the lemma.

Our variational problem is thus to maximize, by varying the boundary $R(\theta)$,

$$\frac{rC}{4}\int_0^{2\pi} |\sin\theta| R(\theta) d\theta, \tag{26}$$

subject to a fixed cocoon energy E_c. The Euler-Lagrange equation is thus found to be

$$\frac{rC}{4}|\sin\theta| + \frac{\nu}{2}C^2 r^3 R(\theta)\int_0^{2\pi} \operatorname{sgn}(\sin\phi)\mathcal{F}(R(\theta), R(\phi), \theta-\phi)d\phi, \tag{27}$$

where

$$\begin{aligned}\mathcal{F}(x,y,\psi) = \frac{1}{2}y^2\log(x^2+y^2-2xy\cos\psi) - xy\cos\psi - \frac{1}{2}y^2 \\ -\frac{1}{2}x^2\cos 2\psi \log\frac{x^2+y^2-2xy\cos\psi}{x^2} + x^2\sin 2\psi\tan^{-1}\left(\frac{y-x\cos\psi}{x\sin\psi}\right) \\ +x^2\sin 2\psi \tan^{-1}(\cot\psi)\end{aligned} \tag{28}$$

Here ν is the Lagrange multiplier. We may write this as

$$|\sin\theta| + b\int_0^{2\pi} \operatorname{sgn}(\sin\phi)\mathcal{F}(R(\theta), R(\phi), \theta-\phi)d\phi, \tag{29}$$

where b is a new multiplier. We may make a substitution $R \to AR$ where A is chosen to make $R(\pi/2) = 1$. (Note that the contributions from the logarithm vanish. We solved the resulting system for $R(\theta$ using the MATLAB routine FSOLVE, assuming symmetry in both the horizontal and the vertical. The result is shown in Figure 3.

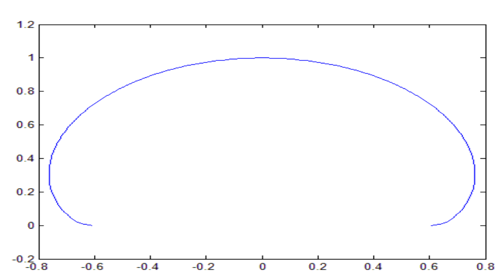

Figure 3. The upper boundary of the optimal vortex configuration.

Computing the energy for this system as $E_0 = \frac{\rho C^3 r}{4} A^4 I_E$, and $U = \frac{rCA}{4} I_U$, we find $I_E \approx 4.24$ and $I_U \approx 3.77$ giving

$$U \approx .93(E_c/\rho)^{1/4}\sqrt{C}\, r^{1/4}. \tag{30}$$

Thus we are lead to propose

Theorem 2 *The cocoon which conserves kinetic energy yields the improved bound for axisymmetric flow without swirl, for large t, given by*

$$\max|\omega_\theta| \le C(C_1 \frac{E_c}{\rho}^{1/4}\sqrt{C}t + C_2)^{4/3}, \tag{31}$$

where $C_1 \approx .7$.

The cocoon energy is defined here by an imagined optimization problem. An acceptable value of E_c, insuring the bound of theorem 2 can be found by simply computing the energy of the support-conserving cocoon at some value of r for which the latter is defined. Since this cocoon is found by a different optimization problem, the energy so obtained will in general be larger than the optimal cocoon energy.

2.4 The filamented cocoon

Since the linear dimension of the cocoon cross section now goes as $r^{-3/4}$, thereby conserving energy, the vorticity support volume *decreases* with r like $r^{-1/2}$. This missing vorticity is not accounted for in the cocoon construction at fixed energy.

The natural next step is therefore to constrain the cocoon by *both* support volume and energy. However, we propose here (and this is the motivation for our third working hypothesis) that this doubly constrained cocoon does not yield a better bound that the cocoon conserving energy alone. The reason is that as $r \to \infty$, vorticity carrying $O(1)$ support volume but zero energy can be deposited in rings arbitrarily close to the plane $z = 0$ containing the core ring. That is to say, in the limit of large r the doubly constrained cocoon is unique, in the sense that arbitrarily nearby bounds are obtained by many extremal, which differ only in the vorticity arbitrarily close to the plane $z = 0$.

This description must be viewed as asymptotic for large r. A significant fraction of the volume (and energy!) can be "left behind" as the energy-conserving cocoon expands. An example of a filamented is an energy-constrained cocoon having volume $K_c r^{-1/2}$ plus the following vorticity distribution: Let $r = r_c$ be the radius of the core ring. Then for $r_1 < r < r_c - k r_c^{-3/4}$

$$\omega_\theta = \begin{cases} -Cr, & \text{for } 0 < z < \frac{1}{8\pi} K_c r^{-5/2}, \\ +Cr, & \text{for } -\frac{1}{8\pi} K_c r^{-5/2} < z < 0. \end{cases} \tag{32}$$

Here k is a constant yielding the left intersection of the cocoon with the plane $z = 0$. The cocoon volume is V_c satisfies $dV_c/dt = -\frac{1}{2} K_c r_c^{-3/2} dr_c/dt$. The flux of volume aft of the cocoon is then $-2\pi r_c H dr_c/dt$ where H is the filament thickness, see Figure 4. Equating these we get $H = O(r^{-5/2})$. Then volume it then being added to the filament at the rate it is being lost by the cocoon. The filament contributes negligibly to both U and to the cocoon energy, so the estimate of theorem 2 remains.

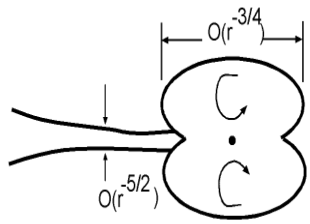

Figure 4. Example of a filamented cocoon with lost support volume extending aft of a cocoon advancing to the right.

2.5 A doubly constrained cocoon in a thin-layer model

The doubly-constrained cocoon is difficult to analyze explicitly in axisymmetric flow without swirl. In the present section we introduce a model where it can be treated fairly directly. The model depends upon the adoption of a thin-layer approximation. This approximation is distinct from *thin-layer Euler dynamics*, which is equivalent to an inviscid version of Prandtl's boundary-layer equations. Rather, we regard the layer as geometrically thin for the purpose of construction of the cocoon and calculation of the energy, but otherwise disregard thinness, in particular in the calculation of U. We shall see below that an optimal cocoon constructed within the model is not geometrically thin, so the model is not consistent as an asymptotic theory. It is simply a model problem where the dual constraints of volume and energy can be studied simultaneously.

We shall consider only the asymptotic cocoon for large r, so the analysis is local and two-dimensional. We show in the appendix that if $\omega_\theta/r = -C$ in the 2D layer $0 < y < Y(x)$, where $-L < x < L$ (we assume symmetry with respect to $x = 0$), and equals C in the layer obtained by reflection in $y = 0$, then the energy of thin year is given approximately by

$$E_c = \frac{2\pi\rho C^2 r^3}{3} \int_{-L}^{L} Y^3(x)dx. \tag{33}$$

For the thin layer our volume constraint is now

$$V_0 = 4\pi r \int_{-L}^{L} Y(x)dx, \tag{34}$$

and we wish to maximize

$$U = \frac{Cr}{4} \int_{-L}^{L} \log\left[\frac{x^2+Y^2}{x^2}\right] dx. \tag{35}$$

We first consider maximization of U subject only to the energy constraint, disregarding the volume constraint. Variation of y_1, y_2 separately yields the Euler-Lagrange equations

$$\frac{Y}{x^2+Y^2} = bY^2, \tag{36}$$

where b is a multiplier. We now represent the cocoon boundary as $x \pm X(y)$ where

$$X(y) = \sqrt{b^{-1}y^{-1} - y^2}, 0 < y < b^{-1/3}. \tag{37}$$

Thus $L = \infty$. To satisfy the energy constraint we note that now

$$E_c = 2\pi\rho C^2 r^3 \int_{A_0/2} y^2 dxdy = \pi\rho C^2 r^3 \int_0^{b^{-1/3}} y^2 X(y)dy$$

$$= \pi b^{-4/3} \rho C^2 r^3 I_E,\ I_E = \int_0^1 z^{3/2}\sqrt{1-z^3}dz \approx .28. \tag{38}$$

Thus the constraint is satisfied by making b proportional to $r^{9/4}$. This implies that the vortical domain is actually $O(r^{-3/4}) \times O(r^{-3/4})$ in dimension. This does not define a thin domain, so the result is not consistent with the slenderness we built into the model. This result is however entirely analogous to that of section 3.

We note that for this extremal

$$U = \frac{Crb^{-1/3}}{2} I_U,\ I_U = \int_0^1 \tan^{-1}(z^{-3/2}\sqrt{1-z^3})dz \approx 1.12. \tag{39}$$

Eliminating b from the expressions for E_0 and U,

$$U = 2^{-1}(2\pi)^{-1/4}(E_c/\rho)^{1/4}\sqrt{C} I_U I_E^{-1/4}\ r^{1/4} \approx .49(E_c/\rho)^{1/4}\sqrt{C}\ r^{1/4}. \tag{40}$$

Thus we again get a bound on vorticity as in theorem 2.

We next consider constraints on both volume and energy, leading to the equation

$$\frac{Y}{x^2+Y^2} = a + bY^2, \tag{41}$$

involving the additional multiplier a. We want to show that the acceptable Y so defined cannot satisfy both energy and volume constraints simultaneously. We now have

$$X(y) = \sqrt{y/(a+by^2) - y^2}. \tag{42}$$

Here $a, b > 0$ and $0 < y < y_m$ where y_m is the unique positive zero of $X(y)$. We then have

$$V_0 = 8\pi r b^{-2/3} \int_0^{z_m(\lambda)} \sqrt{z/(\lambda + z^2) - z^2} dz, \tag{43}$$

$$E_c = 4\pi\rho C^2 r^3 b^{-4/3} \int_0^{z_m(\lambda)} z^2 \sqrt{z/(\lambda + z^2) - z^2} dz, \tag{44}$$

where $z_m = y_m b^{1/3} y_m$ and $\lambda = ab^{-1/3}$. For large r and fixed E_0, V_0, λ, we see that we cannot choose λ, b to satisfy both constraints. The same conclusion is reached when λ is taken as small or large compared to 1. We conclude that we do not find an acceptable extremal preserving both volume and energy.

Again, volume conservation can be viewed as satisfied by filaments which are extensions of either or both of the "tails" of the cocoon.

3 Concluding remarks

By focusing here on the case of axisymmetric flow without swirl, we can see in rather simple terms how the geometrical constraints play a role in estimations of the growth of vorticity in Euler flows. The high degree of symmetry in this case imposes restrictions not present in 3D flows generally. In particular the kinematic cocoon is of constant volume is unable to conserve its kinetic energy. One could imagine a situation where the energy would be exchanged between different vortex structures, but this approach defeats the search for coherent vortex configurations producing singularities.

Once one breaks the constraints of the symmetric flow, however, conservation of energy in a local sense become less of a factor, since energy can be transferred *along* the line of the cocoon. If, for some reason, it turned out that axisymmetric flow without swirl allowed Euler blow-up, then necessarily the blow-up would produce infinite vorticity on an entire ring. In paper II we consider the possibility of a point singularity using a "dynamic" cocoon derived from Euler's equations.

A Energy in a thin layer

We first consider the energy expression needed in the cocoon appropriate to axisymmetric flow without swirl.

We shall establish the form of the energy in two ways, for a more general cocoon of the form $0 < y_1(x) < y < y_2(x)$. We shall show that the cocoon energy is then given by

$$E_c = \frac{\pi\rho C^2 r^3}{3} \int_{-L}^{L} (y_2 - y_1)^2 (2y_1 + y_2) dx. \tag{45}$$

We First we note that

$$E_c = \frac{\rho C^2 r^3}{2} \int_{A_0/2} \int_{A_0'/2} \log \frac{(x-x')^2 + (y+y')^2}{(x-x')^2 + (y-y')^2} dydzdx'dy' \tag{46}$$

Since y, y' are confined to a thin layer, the x, x' integrations may be computed locally. Using

$$\int \log(x^2 + y^2)dx = x\log(x^2 + y^2) - 2x + 2y\tan^{-1}(x/y) \tag{47}$$

we obtain

$$E_c = \frac{\rho C^2 r^3}{2} \int_{-L}^{L} \int_{y_1}^{y_2} \int_{y_1}^{y_2} K(y, y')dydy'dx, \tag{48}$$

where

$$K(y, y') = \begin{cases} 2\pi y' & \text{if } y > y', \\ 2\pi y & \text{if } y < y'. \end{cases} \tag{49}$$

We thus obtain (45).

A second derivation makes use of thin layer approximations in solving a Poisson problem. We note that

$$E_c = \rho\pi C^2 r^3 \int_{A_0/2} \phi(x, y)dxdy \tag{50}$$

where

$$\phi_{yy} \approx \begin{cases} -1 & \text{if } y_1 < y < y_2, -L < x < L, \\ +1 & \text{if } -y_2 < y < -y_1, -L < x < L. \end{cases} \tag{51}$$

We require that both ϕ and ϕ_y be continuous at $\pm y_1$, and that $\phi_y = 0$ at $y = y_2$. The last condition insures that the solution within the layer matches to an external harmonic field.

The solution of the Poisson problem is easily seen to be, for $y_1 < y < y_2$,

$$\phi = -\frac{1}{2}y^2 + y_2 y - \frac{1}{2}y_1^2. \tag{52}$$

Integrating over $A_0/2$ again yields (45).

We remark that if $y_2 - y_1$ is held fixed as a function of the x, the energy is minimized by setting y_1=0, confirming in the thin-layer model the assertion of section 3 that the optimal cocoon under the energy constraint is starlike with respect to the core ring.

Next, suppose that vorticity is confined to a thin layer $0 < z < H(x, y)$ ajacent to $z = 0$ and there has the form $(\omega_x, \omega_y \omega_z) = (\xi(x, y), \eta(x, y), -z(\xi_x + \eta_y))$. From the form of (22) we see that

$$E = \rho \int \omega \cdot \mathbf{B} dV, \ \nabla^2 \mathbf{B} = -\frac{1}{2}\omega. \tag{53}$$

Using the second method described above, we see that in a thin layer

$$\mathbf{B} \approx (H(x,y)z - \frac{1}{2}z^2)(\xi(x.y), \eta(x,y), 0), \tag{54}$$

where we neglect terms which are $o(1)$ in H/L, where L is the scale of variation of ξ, η.

We thus see that to first order

$$E = \frac{\rho}{6} \int \int (\xi^2 + \eta^2) H^3 dxdy. \tag{55}$$

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