Complex Variables II PROBLEM SET 10 (Final problem set) Due April 24, 2007

1. Verify from the integral that $\Gamma(1/2) = (-\frac{1}{2})! = \sqrt{\pi}$. Using only the analytic continuation of Γ following from $\Gamma(z+1) = z\Gamma(z)$, show that

$$\Big(\frac{1}{2}\Big)! = \frac{1}{2}\sqrt{\pi},\ \Big(\frac{3}{2}\Big)! = \frac{3}{2}\frac{1}{2}\sqrt{\pi},\ \Big(-\frac{3}{2}\Big)! = -2\sqrt{\pi},\ \Big(-\frac{5}{2}\Big)! = \frac{4}{3}\sqrt{\pi}.$$

2. (a) Show that the residue of $\Gamma(z)$ at $z = -n$ is $(-1)^n/n!$.

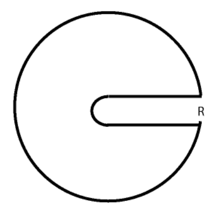
(b) Show from the representation given in equation (97) where C is the contour in figure 3(a) of the lecture notes that if $\mathrm{Re}\, z > 0$ then we obtain

$$\Gamma(1+z) = \int_0^\infty t^z e^{-t} dt$$

and in particlar $\Gamma(n+1) = n!$ at the positive integers.

3. Show, by taking a contour as shown in the figure, that, if $|\mathrm{Re} z| < 1$, then

$$\int_0^\infty \frac{u^z}{(1+u)^2} du = \frac{\pi z}{\sin \pi z}.$$



Be sure to estimate the integral over $|z| = R$.

4. From equation (105) of the lecture notes with $u = v = z$ we have, if $\mathrm{Re}\, z > -1$,

$$\frac{z!z!}{(2z+1)!} = \int_0^1 t^z (1-t)^z dt \ \ (*).$$

(a) Using the substitutions $2t = 1 + s$ and $u = s^2$ show that the integral on the right of (*) may be written $2^{-2z-1}\int_0^1 (1-u)^z u^{-1/2} du$.

(b) Using equation (105) on the last result and simplifying, show that

$$z!(z-1/2)! = 2^{-2z-1}\sqrt{\pi}(2z)!,$$

provided that $\mathrm{Re}\, z > -\frac{1}{2}$.

5. If z is not a negative integer, show that the infinite series

$$\sum_{m=1}^{\infty} R_m,\ R_m = \Big[z \log \frac{m+1}{m} - \log \frac{m+z}{m}\Big]$$

is absolutely convergent. (Hint: expand the logs in series and show that $|R_m| = O(m^{-2}),\ m \to \infty$.)

6. Compute $\zeta(-3)$ from the extension given by equation (130) of the notes.