

1.(a) The flux of fluid through an arc C connecting two streamlines $\psi = a$ and $\psi = b$ is defined as

$$Q = \int_C \vec{q} \cdot \vec{n} ds.$$

Here $\vec{n} = (n_x, n_y)$ is the unit normal to C , as shown in figure (1) below. Then $(-n_y, n_x)$ is the unit tangent to C in the direction from A to B . Show that $Q = b - a$. That is, the change in value of the streamfunction going from one streamline to another is equal to the flux of fluid between the two streamlines.

(b) Problem 2, page 386 of text. Assume that $p + \frac{1}{2}\rho q^2$ is constant over the flow.

2. Consider the flow in the region $r > 0, < \theta < \pi/3$ having complex potential $w = z^3$. From the streamfunction of the flow sketch several of the streamlines.

3. (cf. problem 9 page 387). Consider the transformations $Z = z^2, \zeta = (Z + \frac{1}{Z})$, and the complex potential $w = A\zeta$ where A is a real constant. Describe this as a flow in the region $r \geq 1, 0 \leq \theta \leq \pi/2$. Write expression for the velocity components (u, v) and ψ . Obtain the variation of speed q of the boundary of the region, and verify that $\psi = 0$ there.

4. Obtain the flow over the edge of a flat plate as shown in figure (2), by considering the map $\zeta = \sqrt{z}$, the plate being a branch cut for the square root function. Taking the complex potential to be $w(z) = -i\zeta(z)$, find an expression for the streamfunction and show that the streamlines form a family of parabolas.

5. (a) Prove the following *circle theorem*. Let $w(z)$ be the complex potential of an incompressible potential flow, regular in the disc $|z| \leq a$. Suppose a rigid circular cylinder of radius a is placed in this flow, so that the given flow has to pass around it, i.e. so that the surface of the cylinder becomes a streamline. Then the complex potential for the flow around the cylinder, defined in $|z| > a$, is

$$W(z) = \Phi + i\Psi = w(z) + \bar{w}\left(\frac{a^2}{\bar{z}}\right).$$

Prove this by showing that the streamfunction Ψ vanishes on $a^2 = z\bar{z}$ and that the only singularities of W in $|z| > a$ are those of w .

(Continued on back)

(b) Apply this result to compute the flow field due to a source of strength Q on the x -axis at the point $z_0 = b > a$, having complex potential $w = \frac{Q}{2\pi} \log(z - z_0)$ in the presence of a circular cylinder of radius a and center at the origin. (See figure (3).) Show (using $\overline{\log z} = \log \bar{z}$, that the complex potential is

$$W = m \log(z - b) + m \log(z - b') - m \log z + \log b, \quad m = \frac{Q}{2\pi}, \quad b' = a^2/b.$$

Then show that

$$q = |W'(z)| = m \left| \frac{z^2 - a^2}{(z - b)(z - b')z} \right|.$$

6. Using the Bernoulli theorem in the form $p + \frac{1}{2}\rho q^2 = 0$, and Blasius' theorem in the form

$$X - iY = \frac{1}{2}i\rho \oint_{|z|=a} (W')^2 dz,$$

compute the force (X, Y) on the cylinder in the presence of a source from the above expression for q . Show, by computing residues at $z = 0$ and $z = b'$ that this force is directed from the center of the cylinder toward the source and has magnitude

$$\frac{2\pi\rho m^2 a^2}{b(b^2 - a^2)}.$$

