Complex Variables II PROBLEM SET 7 Due March 20, 2007

1. (a) If $u(r,\theta)$ is a harmonic function for $r < R$, verify from the results of section 117 that u satisfies the *mean-value property*

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\phi)d\phi,\ r < R.$$

(b) Show that u also satisfies

$$u(0) = \frac{1}{\pi r^2}\int\int_{\rho<r} u(\rho,\phi)dA,$$

where dA is the area element.

2. Using the Fourier series representation on $-\pi, +\pi$, show that the harmonic function defined in $r < 1$ which has the boundary values $u(1,\phi) = \phi/2, -\pi < \phi < \pi$ is equal to $\mathrm{Im}\big(Log(1+z)\big)$.

3. Suppose that u is harmonic and *positive* in the disk $0 \leq r \leq r_0$ and represented there by the Poisson integral formula. Show that then

$$u(0)\Big(\frac{r_0 - r}{r_0 + r}\Big) \leq u(r,\theta) \leq u(0)\Big(\frac{r_0 + r}{r_0 - r}\Big).$$

4. Problem 4, page 425 of text.

5. Problem 1, page 431 of text.

6. Problem 4, page 436 of text. Note: The heat flux into the plate along the segment $0 < x < 1$ is equal to $-\partial T/\partial y(x,0)$, and the heat flux out of the plate on the positive y-axis is $\partial T/\partial x(0,y)$.

7. By finding a suitable function analytic in the upper half-plane, show that the Hilbert transform of $\frac{\sin x}{x}$ is $\frac{1-\cos x}{x}$.

8. (a) Suppose we wish to solve the Neumann problem for the disk $r \leq r_0$ using Fourier series. We know any harmonic function in $r < r_0$ has an expansion

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(\frac{r}{r_0}\big)^n(a_n\cos n\theta + b_n sinn\theta).$$

Show that if we want $\frac{\partial u}{\partial r}\big|_{r=r_0} = V(\theta)$, we should take

$$(a_n, b_n) = \frac{r_0}{\pi n}\int_0^{2\pi} V(\phi)(\cos n\phi, \sin n\phi)d\phi.$$

(b) Use (a) to construct a harmonic function u in the unit circle having $\frac{\partial u}{\partial r}\big|_{r=1}$ equal to 1 for $-\pi/2 < \theta < \pi/2$ and -1 for $\pi/2 < \theta < 3\pi/2$.