1. Complete all the steps of the proof of lemma 3, chapter 1, of the lecture notes for lectures 9-14.

2. Verify for step (iii) of the proof of the mapping theorem in the lecture notes:

$$G'(\zeta) = |H'(\zeta)| = \frac{1+|\alpha|}{2\sqrt{|\alpha|}} F'(\zeta) > F'(\zeta).$$

3. If $f(z)$ is analytic and $|f(z)| \le 1$ for $|z| < 1$, ad $\alpha| < 1$, show that

$$\Big| \frac{f(z) - f(\alpha)}{1 - \overline{f(\alpha)} f(z)} \Big| \le \Big| \frac{z-\alpha}{1-\bar{\alpha}z} \Big|.$$

Hint: $\zeta = Tz = \frac{z-\alpha}{1-\bar{\alpha}z}$ maps the unit disk onto itself carrying α to the origin. Also $Sw = \frac{w-f(\alpha)}{1-\overline{f(\alpha)}w}$ maps the unit disk onto itself carrying $f(\alpha)$ to the origin. Then show that $g(\zeta) = Sf(T^{-1}\zeta)$ satisfies the assumptions of the Schwarz lemma.)

4. Show that, if x is a positive real variable, $f(x) = (1+x)^{-1}$ and $g(x) = (1 + e^{-x}(1+x)^{-1}$ have the same asymptotic approximatioin $h_N(x)$ with the property that $f, g - h_N = O(x^{-(N+1)}), x \to \infty$.

5. Find a few terms of the asymptotic expansion of

$$\int_x^\infty e^{-t^4} dt, \; x \to 0.$$

Your answer should be a series in power of x up to and including x^9. Here x, t are real. Note that $\int_0^\infty e^{-t^4} dt$ may be expressed in terms of the gamma function $\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} du$. What is the order of the first neglected term?

6. Use the method of stationary phase to find the leading, dominant term in the asymptotic expansion of

$$\int_0^1 e^{it(\lambda - \lambda^3)} d\lambda,$$

as $t \to +\infty$. Here t, k are real variables.