1. (a) Find $\lim_{z\to i}(z+1/z)$. (b) Show that $\lim_{z\to z_0} f(z)g(z) = 0$ if $\lim_{z\to z_0} f(z) = 0$ and there exists a positive number M such that $|g| \le M$ in some ϵ-neighborhood of z_0.

2. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and$g'(z_0) \neq 0$ exist. Show that

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

3. Using the differential operators $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ show that $\frac{\partial \bar{f}}{\partial \bar{z}} = \overline{\frac{\partial f}{\partial z}}$, where $f = u(x,y) + iv(x,y)$ and all first partials of u and v exist.

4. Using the Cauchy-Riemann conditions: (a) show that $f = 2x + ixy^2$ and $f = e^x e^{-iy}$ are differentiable nowhere. (b) Determine for what complex constants a, b, if any, the function $f = x^2 + axy + by^2$ is differentiable everywhere. (c) Determine for what points (x, y) the function $f = (x - iy)(2 - x^2 - y^2)$ has a derivative.

5. Expressing $f' = u_x + iv_x$ in terms of polar coordinates and using the polar form of the Cauchy-Riemann conditions, show that if f is differentiable at $z_0 = re^{i\theta_0}$ then

$$f'(z_0) = e^{-i\theta}(u_r + iv_r) = \frac{-i}{z_0}(u_\theta + iv_\theta)$$

where the partials $u_r, v_r, u_\theta, v_\theta$ are evaluated at (r_0, θ_0).

6.*(A star on a problem indicates that it has been asked on the CIMS written examination in complex variables.) Let $f(z)$ be holomorphic on an open set S. Show that, if $|f(z)| = 1$ on S, then f is a constant on S. Answer this in the following steps:
(a) Verify that $\frac{\partial \bar{f}}{\partial z} = 0$ if f is an analytic function of z in S, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$.
(b) Use $|f|^2 = f\bar{f}$.
(c) Prove (6) in another way by differentiating $u^2 + v^2$ with respect to x, y.