Complex Variables I PROBLEM SET 3 Due October 3, 2006

1. (a) Find all values of z such that $e^{z-1} = 1 + i$.

(b) Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in some domain D. State why the functions $U(x, y) = e^{u(x,y)} \cos v(x, y)$, $V(x, y) = e^{u(x,y)} \sin v(x, y)$ are conjugate harmonic functions on D. (You may use the fact that the real and imaginary parts of any analytic function are conjugate harmonic.)

2. a) Show that $\text{Log}\,(1 - i) = \frac{1}{2} \ln 2 - \frac{\pi}{4} i$; that $\log\, i = \left(\frac{1}{2} + 2n\right)\pi i$, n any integer.

(b) Show that the set of values of $\log\,(i^{1/2})$ is $(n + 1/4)\pi i$ (n an integer) and that the same is true of $\frac{1}{2} \log\, i$. Show that the set of values of $\log(i^2)$ is *not* the same as the set of values of $2 \log\, i$.

3.* Find the real and imaginary parts of (a) $(1 + i)^i$; (b) of i^z, $z = x + iy$.

In (4) and (5) below recall $\sin z = \sin x \cosh y + i \cos x \sinh y$, $\cos z = \cos x \cosh y - i \sin x \sinh y$.

4. Using $\sin^2 x + \cos^2 x = 1$ and $\cosh^2 y - \sin^2 y = 1$ show that (a) $|\sin z|^2 = \sin^2 x + \sinh^2 y$, (b) $|\cos z|^2 = \cos^2 x + \sinh^2 y$. Then show that (c)$|\sin z|^2 + |\cos z|^2 \geq 1$, with equality if and only if z is real.

5. If $z = x + iy$ and $\tan z = Re^{i\theta}$ ($R > 0, \theta$ real), show that

$$\tan\theta = \frac{\sinh 2y}{\sin 2x}.$$

6. We define $w = \tan^{-1} z$ as equivalent to $z = \frac{\sin w}{\cos w}$. Using this and the definitions of $\sin z, \cos z$ to show that

$$\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1 + iz}{1 - iz}\right).$$

Use this formula to show that

$$\tan^{-1} z_1 + \tan^{-1} z_2 = \tan^{-1}\left(\frac{z_1 + z_2}{1 - z_1 z_2}\right).$$