Complex Variables I PROBLEM SET 4 Due October 10, 2006

1. (a) Evaluate $\int_1^2 \left(\frac{1}{t} - i\right)^3 dt$.

(b) Verify that $\frac{d}{dt}\left(\frac{e^{izt}}{iz}\right) = e^{izt}$, $z \neq 0$. Use this to evacuate $\int_{-1}^{+1} e^{izt} dt$.

2.

(a) Let $w(t)$ be a continuous complex-valued function of t defined on the interval $a \leq t \leq b$. By considering he special case $w = e^{it}$ on the interval $0 \leq t \leq 2\pi$, show that the mean value theorem for definite integrals is calculus does *not* apply in the complex case. That is, show that it need not be true that there is a real number c in $a \leq t \leq b$ such that

$$\int_a^b w(t)dt = w(c)(b-a).$$

(b) Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t), a \leq t \leq b$. Show that if $w(t) = f[z(t)]$ then

$$w'(t) = f'[z(t)]z'(t).$$

(Hint: Write $f(z) = u(x,y) + iv(x,y)$ an $z = x(t) + iy(t)$, so that $w(t) = u[x(t),y(t)] + iv[x(t),y(t)]$. Then compute $w'(t)$ using the chain rule and use the Cauchy-Riemann equations.)

3. Show that

(a) $\int_C \frac{z+2}{z} dz = 4 + 2\pi i$ when C is the semicircle $z = 2e^{i\theta}$, $(\pi \leq \theta \leq 2\pi)$.

(b) $\int_C \pi \exp(\pi \bar{z}) dz = 4(e^\pi - 1)$ when C is the boundary of the square with vertices at the points $0, 1, 1+i, i$ and the orientation of C is in the counterclockwise direction.

4. Evaluate $\int_C$ Log z dz where the principal branch Log z is taken and C is $z = 2e^{it}, -\pi/2 \leq t \leq \pi/2$. Log $z = \ln r + i\Theta, -\pi < \Theta < \pi$.)

(a) Do this first as an integral with respect to t.

(b) Then do it using the antiderivative $F(z) = z\text{Log}z - z$, first verifying that this is indeed an antiderivative of Log$z$

5. Let C be an arc of the circle $|z| = R (R > 1)$ of included angle $\pi/3$. Show that

$$\Big|\int_C \frac{dz}{z^3+1}\Big| \leq \frac{\pi}{3}\Big(\frac{R}{R^3-1}\Big),$$

and deduce that $\lim_{R\to\infty} \int_C \frac{dz}{z^3+1} = 0$.

6. Evaluate, using antiderivatives:

(a) $\int_i^{i/2} z e^{2z^2} dz$ (b) $\int_0^{\pi+2i} \sin(z/2) dz$ (c) $\int_i^2 \frac{dz}{z^2}$ (any smooth arc avoiding the origin).