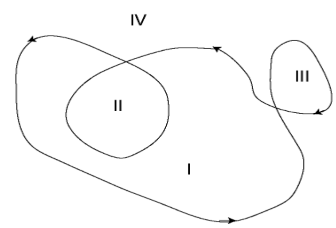
Complex Variables I PROBLEM SET 5 Due October 17, 2006

1. Problem 2, page 153 of text. (Recall, if f is analytic within the region bounded by two simple, closed, positively oriented contours, and is also analytic on the contours, then the integral of f is the same on each contour.)

2.* Evaluate

$$I(z_0) = \int_C \frac{dz}{z - z_0}$$



taken once around the contour shown in the figure, for all point z_0 not on the contour. Be sure to consider z_0 located in each of the regions I, II, III, IV.

3. Problem 4, page 154.

4. If $P(z)$ is the polynomial $\Pi_{k=1}^{n}(z - z_k)$ where the z_k are complex numbers, and C is a simple closed contour containing all of the z_k, show that

$$\int_C \frac{P'(z)}{P(z)} dz = 2\pi i n.$$

5. By using the method and the rectangular contour of problem (3) above, applied now to the function $f(z) = \frac{1}{1+z^2}$, show that

$$\int_{-\infty}^{+\infty} \frac{(1 - b^2 + x^2)}{(1 - b^2 + x^2)^2 + 4b^2x^2} dx = \pi,$$

where b is real and $0 < b < 1$. (Recall $\int \frac{1}{1+x^2} dx = \tan^{-1} x$. Be sure to recognize integrals of functions which vanish by being odd in x. Also note that $|1 + a^2 - y^2 + 2iay| \geq |1 + a^2 - y^2| \geq 1 + a^2 - b^2 > 0$ on the integrals on the vertical sides.)

6.* Evaluate

$$\int_0^\infty \frac{\sin x}{\sqrt{x}} dx$$

By integrating $\frac{e^{iz}}{\sqrt{z}}$ over the contour $C = C_1 + C_2 + C_3 + C_4$ shown in the figure, involving arcs of radius ϵ and R, then letting $\epsilon \to 0$ and $R \to \infty$.(Hint: To estimate the integral C_2 use $\sin\theta \geq 2\theta/\pi, 0 \leq \theta \leq \pi/2$. You should show that C_4 vanishes as $\epsilon \to 0$ and that C_2 vanishes as $R \to \infty$. Take $\sqrt{z} = \sqrt{r}e^{i\theta/2}, 0 \leq \theta \leq \pi/2$. You can check your result against $\int_0^\infty \sin(t^2)dt$ as done in class by a change of variables.)