Complex Variables I **Review Problems**

1. Find all values of $\sqrt{1-\sqrt{3}i}$, of $log(\sqrt{i})$, and of i^i.

2. Sketch the region $\mathrm{Re}(1/z) \leq 1/2$.

3. Which of he following are analytic functions of z: $f = x^2 + iy^2$, $f = x^2 - y^2 + i2xy$, $e^y(\cos x + i \sin x)$.

4. Let f be an entire function such that $|f| \leq R$ on $|z| = R$, $R > 0$ arbitrary. Prove that $f = az + b$ for some complex numbers a, b.

5. (a) Evaluate

$$\oint z^2 \sin(1/z) dz$$

taken once in the positive sense around $|z| = 1$.

(b)Evaluate

$$\oint_C \frac{\cosh z}{z \sinh^2 z} dz$$

where C is the circle $|z - i| = 4$ with positive orientation.

6. Find the Laurent series about $z = 0$ valid in the annulus $1 < |z| < 2$, for the function $f = \frac{1}{(z+1)(z+2)}$.

7. By summing, determine the function $f(z)$ with the Laurent expansion

$$f(z) = \sum_{n=0}^{\infty} (z/2)^n + \sum_{n=1}^{\infty} (1/z)^n.$$

What is the Laurent expansion of $f(z)$ in $|z| > 2$? What is the series, in powers of $z - i$, which converges to f in the disk $|z - i| < \sqrt{2}$?

8. If a is a complex number such that $Re(a) > 1$, show that

$$\frac{2}{1-z^2} = \sum_{n=1}^{\infty} (-1)^n \frac{(a-1)^{n-1}}{(z-a)^n} + \sum_{n=0}^{\infty} (-1)^n \frac{(z-a)^n}{(a+1)^n}$$

for all z satisfying $|a - 1| < |z - a| < |a + 1|$.

9. Show that, is $|z| < 1$,

$$\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n.$$

10. Let $f(z)$ be analytic everywhere except possibly at z_0 . Describe precisely what is meant by the following additional properties:

(a) f has a residue 2 at z_0.
(b) f has an essential singularity at z_0.
(c) f has a removable singularity at z_0.
(d) f has a pole of order 7 at z_0.
(e) f is entire.

11. Using residue theory, evaluate $\int_0^\infty \frac{1}{1+x^3}$. (Your contour should include a portion of $\theta = 2\pi/3$.)

12. Let $P(z) = a_0 + a_1 z + \ldots + a_n z^n$, $a_n \neq 0$ be a polynomial of degree n, and $Q(z) = b_0 + b_1 z + \ldots + b_m z^m$, $b_m \neq 0$ is a polynomial of degree $m \geq n+2$. Show that, if all the zeros of $Q(z)$ are interior to the simple closed contour C, then

$$\int_C \frac{P(z)}{Q(z)} dz = 0.$$

13. Using residue theory, show that

$$\int_0^\infty \frac{\ln x}{x^2 + a^2} dx = \pi \frac{\ln a}{2a}, \quad a > 0.$$

(use an indented contour.)

14. Problem 5, page 265 of text.

15.Problem 5 page 276 of text.

16. Suppose the function $f(z)$ is analytic for $|z| < 2$. Show that, for ϵ sufficiently small, $z - \epsilon f(z)$ has exactly one solution in $|z| < 1$. Suppose you know additionally that $|f(z) \leq 5$ on $|z| = 1$. How small must ϵ be to insure that f has exactly one solution in $|z| < 1$?

17. Find the number of roots (counting multiplicities) of $z^8 - 4z^5 + z^2 - 1 = 0$ lying in the disk $|z| < 1$. How many roots lie in the disk $|z| < 2|$?