ANSWERS TO REVIEW PROBLEMS

1. • $1-\sqrt{3}i = e^{-\pi/3 i}$, so $\sqrt{1-\sqrt{3}i} = \pm e^{\frac{\pi}{6}i} = e^{\frac{5\pi}{6}i}, e^{-\pi/6 i}$
   • $\log\sqrt{i} = \log e^{i\pi/4}, \log e^{i\frac{5\pi}{4}} = i\frac{\pi}{4} \pm 2\pi i n, i5\frac{\pi}{4} \pm 2\pi i n, n=0,1,2,..$
   • $i^i = e^{i\log i} = e^{i(\frac{\pi}{2}i \pm 2\pi n i)} = e^{-(\pi/2 \pm 2\pi n)}$.

2. $Re\left(\frac{1}{z}\right) = \frac{x}{r^2} < 1/2 \iff \frac{1}{2}x^2 - x + \frac{1}{2}y^2 > 0$ or $\frac{1}{2}(x-1)^2 + \frac{1}{2}y^2 > \frac{1}{2}$

So region is exterior of circle of radius 1 centered at $(1,0)$.

3. $u = x^2$, $\nabla^2 u = 2 \neq 0 \Rightarrow f$ not analytic
$f = x^2 - y^2 + 2ixy$ satisfies C.R. equations $\Rightarrow$ analytic
$f = e^y(\cos x + i\sin x)$ does not satisfy $u_x = v_y \Rightarrow$ not analytic.

4. Let z_0 be any point. Now for R sufficiently large $|z| \leq R$ will contain z_0, so that by the Cauchy formula for derivatives

$$f''(z_0) = \frac{1}{2\pi i} 2 \oint_{|z|=R} \frac{f(z)}{(z-z_0)^3}\, dz$$

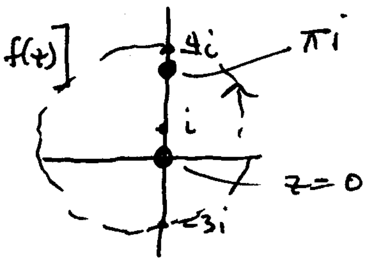
Thus $|f''(z_0)| \leq \frac{\pi}{\pi} \max_{|z|=R} |f| \frac{1}{R^2} \leq \frac{1}{R} \to 0$

as $R \to \infty$. It follows that $f''(z_0) = 0$ for any z_0. Taking anti derivatives, $f = az + b$, a, b being arbitrary complex numbers.

5 (a) $I = \oint_{|z|=1} z^2 \sin\frac{1}{z}\, dz = \oint_{|z|=1} z^2\left(\frac{1}{z} - \frac{1}{6}\frac{1}{z^3} + \cdots\right.$

$\operatorname*{Res}_{z=0} f(z) = -\frac{1}{6}, \qquad I = -\frac{\pi i}{3}$

(b) $I = \oint_{|z-i|=4} \frac{\cosh z}{z \sinh^2 z}\, dz = 2\pi i \left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=\pi i} f(z)\right]$



We have pole of order 3 at origin.

$\sinh z$ has zeros when $e^{2z} = 1$ or $x=0,\ 2y = 2\pi n,\ n = 0, \pm 1, \pm 2.$

So πi is another pole within the contour.

At $z=0$, $\sinh z = z + \frac{1}{6}z^3 + \cdots$, $\cosh z = 1 + \frac{1}{2}z^2 + \cdots$

so $f(z) = \frac{1\,(1+\frac{1}{2}z^2+\cdots)}{z^3(1+\frac{1}{6}z^2+\cdots)} = \frac{1}{z^3}\left(1+\left(\frac{1}{2}-\frac{1}{6}\right)z^2 + \cdots\right)$

$\Rightarrow \operatorname*{Res}_{z=0} f(z) = \frac{1}{3}$

At $z = \pi i$, $\cosh z = -1 + O(z-\pi i)^2$

$$\frac{1}{z \sinh^2 z} = \frac{1}{z - i\pi + i\pi \quad (z-i\pi)^2 + \cdots}$$

$$= \frac{\frac{1}{i\pi}\left(1 - \frac{z-i\pi}{i\pi} + \cdots\right)}{(z-i\pi)^2}$$

So residue is $\frac{1}{\pi^2}$

So $I = 2\pi i\left[\frac{1}{3} + \frac{1}{\pi^2}\right]$

6 $$\frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2} = \frac{1}{z(1+\frac{1}{z})} - \frac{1}{2(1+\frac{z}{2})}$$

so in $1 < |z| < 2$

$$\frac{1}{(z+1)(z+2)} = \frac{1}{z}\sum_{n=0}^{\infty} (-1)^n z^{-n} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n$$

7 $$f(z) = \frac{1}{1-\frac{z}{2}} + \frac{1}{z}\frac{1}{1-\frac{1}{z}}, \quad 1 < |z| < 2$$

$$= \frac{2}{2-z} + \frac{1}{z-1} = \frac{z}{(2-z)(z-1)}$$

Then

$$f(z) = \frac{2}{2-(z-i)-i} + \frac{1}{(z-i)+i-1} = \frac{2}{2-i\left(1-\frac{(z-i)}{2-i}\right)} + \frac{1}{i-1\left[1+\frac{(z-i)}{i-1}\right]}$$

$$= \frac{2}{2-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{2-i}\right)^n \mp \frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n$$

8 $$\frac{2}{1-z^2} = \frac{1}{1-z} + \frac{1}{1+z}$$

$$= \frac{1}{1-a-(z-a)} + \frac{1}{1+a+(z-a)}$$

$$= \frac{-1}{z-a}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a-1}{z-a}\right)^n + \frac{1}{1+a}\sum_{n=0}^{\infty}\left(\frac{z-a}{1+a}\right)^n(-1)^n$$

$$= \sum_{n=1}^{\infty}(-1)^n\frac{(a-1)^{n-1}}{(z-a)^n} + \frac{1}{1+a}\sum_{n=0}^{\infty}\left(\frac{z-a}{1+a}\right)^n(-1)^n$$

whe $|a-1| < |z-a| < 1+a$

9. $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z|<1$$

We can differentiate term by term

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} n z^{n-1} = \sum_{n=1}^{\infty} n z^{n-1}$$

so $$\frac{z}{1-z^2} = \sum_{n=1}^{\infty} n z^n$$

10. (a) The Laurent series at z_0 contains the term $\frac{2}{z-z_0}$.

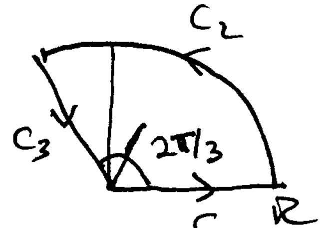
(b) The principal part of the L-series of f at z_0 is infinite

(c) " " " " " " " " " " vanishes

(d) " " " " " " " " " " is of the form $\frac{b_7}{(z-z_0)^7} + \frac{b_6}{(z-z_0)^6} + \cdots$, $b_7 \neq 0$

(e) f is analytic at z_0 as well.

11. $$\lim_{R\to\infty} \int_{C_1} f(z)\,dz = I = \int_0^{\infty} \frac{dx}{1+x^3}$$



where $f(z) = \frac{1}{1+z^3}$. Easy to show

$$\left|\int_{C_2} f(z)\,dz\right| \leq \pi \frac{R}{R^3-1} \to 0 \quad \text{as } R\to\infty.$$

$$\lim_{R\to\infty} \int_{-C_3} f(z)\,dz = e^{2\pi/3\, i}\, I.$$ Thus by Cauchy thm.

$$(1-e^{2\pi/3\, i})\, I = 2\pi i \left(\operatorname{Res}_{z=e^{i\pi/3}} f(z)\right) = \frac{1}{3\, e^{2i\pi/3}}$$

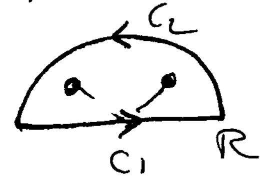
Thus $I = \frac{1}{3}\,\frac{2\pi i}{e^{2\pi i/3} - e^{4\pi i/3}} = \frac{2\pi i}{3 \cdot 2i \sin\frac{2\pi}{3}} = \frac{\pi}{3\sin 120^\circ} = \frac{\pi}{3\sin 60^\circ} = \frac{2\pi}{3\sqrt{3}}$

(12) $I = \int_C \frac{P(z)}{Q(z)}\,dz = \int_C f(z)\,dz$, $f(z)$ analytic outside C

$\Rightarrow I = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2}\,\frac{P(1/z)}{Q(1/z)}$

since degree $Q \geq$ degree of $P + 2$, it follows that $\frac{1}{z^2}\,\frac{P(1/z)}{Q(1/z)}$ has a L- series about $z=0$ valid exterior and on C having zero principal parts, in particular $\operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 0$ and $I = 0$.

(14) $I = \int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\,dx = \operatorname{Im} \int_{-\infty}^{+\infty} \frac{x e^{iax}}{x^4+4}\,dx$, $a > 0$.



Simple poles of the denominator of $\frac{z e^{iaz}}{z^4+4}$

occur at $z^2 = \pm 2i = 2e^{i\pi/2}, 2e^{-i\pi/2}$

$z = \sqrt{2}e^{i\pi/4}, \sqrt{2}e^{5i\pi/4}\ \sqrt{2}e^{-i\pi/4}\ \sqrt{2}e^{3i\pi/4}$

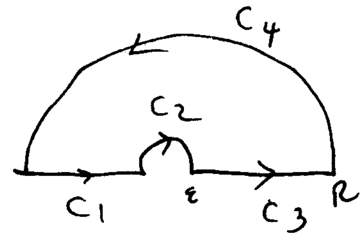
Thus we find $\left|\int_{C_2} f(z)\,dz\right| \leq \frac{\pi}{R}\,\frac{R^2}{R^4-4} \to 0$ as $R \to \infty$

$\lim_{R\to\infty} \int_{C_1} f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=\sqrt{2}e^{i\pi/4}} f(z) + \operatorname*{Res}_{z=\sqrt{2}e^{3i\pi/4}} f(z) \right] = 2\pi i \left[\frac{\sqrt{2}e^{i\pi/4}\, e^{ia(1+i)}}{4e^{3i\pi/4} \cdot 2\sqrt{2}} + \frac{\sqrt{2}e^{3i\pi/4}\, e^{ia(-1+i)}}{2\sqrt{2}\, 4\, e^{9i\pi/4}} \right]$

$$\lim_{R\to\infty} \int_C f(z)\,dz = \frac{2\pi i}{8}\left[-i\, e^{-a+ia} + \frac{i}{8}\, e^{-a-ia} \right]$$

$$= \frac{\pi}{4}\, 2i \sin a\, e^{-a}$$

and $I = \frac{\pi}{2} \sin a\, e^{-a}.$



13 $\displaystyle\int_0^\infty \frac{\ln x}{x^2+a^2}\,dx = I$

$f(z) = \dfrac{\log z}{z^2+a^2}$ $\qquad \log z = \ln r + i\theta \qquad 0 < \theta < 2\pi$

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{C_1+C_3} f(z)\,dz = I - e^{i\pi}\int_0^\infty \frac{\ln r + i\pi}{r^2+a^2}\,dr$$

$$= 2I + i\pi \underbrace{\int_0^\infty \frac{dr}{r^2+a^2}}_{\frac{1}{a}\frac{\pi}{2}}, \quad a>0.$$

Claim $2I + i\pi\left(\frac{\pi}{2a}\right)$

$$= 2\pi i \operatorname{Res}_{z=ia} f(z) = 2\pi i \frac{\ln ia}{2ia} = \frac{\pi}{a}\left(\ln a + i\frac{\pi}{2}\right)$$

Thus $I = \frac{1}{2}\frac{\pi}{a}\ln a.$

* Indeed we have

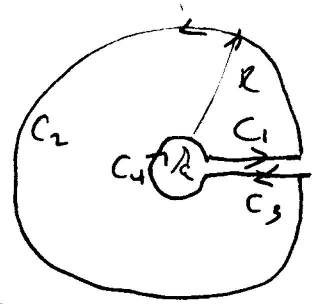
$$\left|\int_{C_4} f(z)\,dz\right| \le \pi \frac{\sqrt{(\ln R)^2 + \pi^2}\, R}{R^2-a^2} \to 0 \quad \text{as } R\to\infty$$

$$\left|\int_{C_2} f(z)\,dz\right| \le \left(\frac{(\ln \varepsilon)^2 + \pi^2}{a^2-\varepsilon^2}\right)\pi\varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$

So our result follows from the residue theorem.

15 $f(z) = \dfrac{z^{1/3}}{(z+a)(z+b)}$. $\quad I = \displaystyle\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}$ $\quad a > b > 0$.

$$= \frac{e^{\frac{1}{3}\log z}}{(z+a)(z+b)}, \quad 0 < \arg z < 2\pi .$$



$$\lim_{R\to\infty} \lim_{\varepsilon\to} \int_{C_1} f(z)\,dz = I$$

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{-C_3} f(z)\,dz = \int_0^\infty \frac{e^{\frac{1}{3}\ln r} e^{2\pi i/3}}{(x+a)(x+b)} = e^{2\pi i/3} I .$$

So if $\displaystyle\lim_{R\to\infty} \left|\int_{C_2} f(z)\,dz\right| = 0$, $\displaystyle\lim_{\varepsilon\to 0} \left|\int_{C_4} f(z)\,dz\right| = 0$

we have $\left(1 - e^{2\pi i/3}\right) I = 2\pi i \left[\operatorname*{Res}_{z=-a} f + \operatorname*{Res}_{z=-b} f\right]$

$$= 2\pi i \left[\frac{b^{1/3} e^{i\pi/3}}{a-b} + \frac{a^{1/3} e^{i\pi/3}}{b-a} \right]$$

$$= 2\pi i \left[\frac{a^{1/3} - b^{1/3}}{b-a} \right] \frac{1}{e^{-\pi/3\, i} - e^{\pi/3\, i}} = \pi \left[\frac{b^{1/3} - a^{1/3}}{b-a} \right] \frac{1}{\sin \pi/3}$$

$$= \frac{2\pi}{\sqrt{3}} \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b} \quad \text{as claimed.}$$

To justify, note

$$\left| \int_{C_2} f(z)\,dz \right| \le \frac{\pi R^{1/3} R}{(R-a)(R-b)} \to 0 \text{ as } R\to\infty$$

$$\left| \int_{C_4} f(z)\,dz \right| \le \frac{\varepsilon^{1/3} \pi \varepsilon}{(a-\varepsilon)(b-\varepsilon)} \to 0 \text{ as } \varepsilon \to 0.$$

16 Since f is analytic within and on the circle $|z|=1$, we know $|f| \leq M$ in $|z| \leq 1$ for some $M > 0$.
Since $|z| = 1$ on $|z|=1$ and $|z| > \varepsilon |f|$ on $|z|=1$ if $\varepsilon < \frac{1}{M}$ $\Rightarrow$ $z + \varepsilon f$ has at most one root in $|z|<1$. $M \geq 5$ $\Rightarrow$ $\varepsilon < \frac{1}{5}$.

17 $|4z^5| = 4$, $|z^8 + z^2 - 1| \leq 3$ on $|z| = 1$
Thus $4z^5$ and $z^8 - 4z^5 + z^2 - 1$
have the same number, 5, of roots in $|z|<1$.

For $|z|<2$, we note $|z|^8 = 256$ on $|z|=2$
$|4z^5 - z^2 + 1| \leq 4 \cdot 32 + 4 + 1 = 133$ on $|z|=2$
so there are 8 roots in $|z| \leq 2$, thus 3 in $1 < |z| < 2$.