1. We asserted in class that $\phi = \frac{\mu}{2}(\partial u_i/\partial x_j - \partial u_j/\partial x_i)^2 - \frac{2\mu}{3}(\nabla \cdot \vec{u})^2$ is non-negative. Prove this.

2. Show that for a perfect gas, another form of the energy equation for a viscous, heat conducting fluid is

$$\rho c_v \frac{DT}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} = \phi + \nabla \cdot k\nabla T.$$

(Hint: Start with $dS = \left(\frac{\partial S}{\partial T}\right)_v dT + \left(\frac{\partial S}{\partial v}\right)_T dv$.)

3. Show that for an inviscid gas with zero heat conductivity, (i.e. $\mu = k = 0$), the energy equation may be written

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot (\vec{u}\rho e) = -p\nabla \cdot \vec{u}.$$

4. (a) Show that

$$c_p - c_v = T\left(\frac{\partial p}{\partial T}\right)_v \left(\frac{\partial v}{\partial T}\right)_p$$

(Hint: Regarding s as a function of T, v, $ds = (\frac{\partial s}{\partial T})_v dT + (\frac{\partial s}{\partial v})_T dv$, from which we can get an expression for $\left(\frac{\partial s}{\partial T}\right)_p$. Now use $T(\frac{\partial s}{\partial T})_p = c_p$, $T(\frac{\partial s}{\partial T})_v = c_v$, and a Maxwell relation.) Show that, for a perfect gas, this relation gives $R = c_p - c_v$.

(b) Show that for a perfect gas

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma RT.$$

Note that this quantity equals c^2 where c is the speed of sound under isentropic conditions.

5. For the study of thermal convection in water, it is usually assumed that the fluid density is a function of temperature alone. The energy equation is then usually approximated as a temperature equation of the form

$$\rho c_p \frac{DT}{Dt} - \nabla \cdot k\nabla T = 0.$$

Justify this as an approximation to the energy equation given in class, namely

$$\rho c_p DT/Dt - \rho T(\partial v/\partial T)_p Dp/Dt = \phi + \nabla \cdot k\nabla T.$$

You should make use of the data for water at 20^oC given on pages 596 and 597 of Batchelor. (Note $\beta = v^{-1}(\partial v/\partial T)_p$ and our k is the same as Batchelor's k_H.) Assume a characteristic fluid speed is $U \approx 1\ cm/sec$, in a fluid layer of thickness $L \approx 1\ cm$. That is, use U, L to estimate terms involving spatial derivatives and velocity. Also take L/U as a characteristic timescale, one $dyne/cm^2$as a characteristic pressure, and 10^oC as typical of T and it's variations. (Note 1 joule= 10^7 dyne-cm.) Note that the heat conduction term is retained, even though relatively small, because it can be important in boundary layers.

6. For a perfect gas, c_v, c_p are functions of T alone and so $e = \int c_v dT$. Also then $c_p - c_v = R =$ constant. Using these facts show that for steady flow of a perfect gas Bernoulli's theorem may be written $\frac{1}{2}u^2 + \int cp(T)dT + \Psi =$ constant on streamlines.