Fluid Dynamics II PROBLEM SET 2 Due February 23, 2004

1. Show that sound waves may be treated using a potential ϕ satisfying the wave equation $\phi_{tt} - c^2\nabla^2\phi = 0$ with velocity $\mathbf{u} = \nabla\phi$ and pressure $p = -\rho_0\phi_t$. Here $c^2 = dp/d\rho(\rho_0)$. Find the potential of the three-dimensional, spherically symmetric sound field emitted by an oscillating sphere. Assume oscillations are of amplitude a about a radius r_0, where $a << r_0$, so the boundary condition to be satisfied is

$$\frac{\partial\phi}{\partial r}(r_0, t) = a\sin\omega t = aIm(e^{i\omega t})$$

approximately. (Why is the last expression approximately correct?) Assume also that only outgoing waves appear in the solution. From your expression for ϕ compute (best to do this in the complex form) the radial velocity field $u_r = \frac{\partial\phi}{\partial r} = \phi_r$. Your expression will involve the dimensionless parameter $\Omega = \omega r_0/c$. (The frequency of the standard note A above middle C is 440 Hertz, or cycles per second, in which case $\omega = 880\pi$. Taking $r_0 = 1cm$ and $c = 350m/sec$ we get $\omega r_0/c \approx .08$. We may take the parameter to be of order unity.) The length scale $R = c/\omega \approx$ 12cm separates the "near" and "far" fields of the sound pattern. Discuss the r-variation of $u_r\phi_r$ in the two cases $r_0 + a < r << R$ cm and $r >> R$.

2.A semi-infinite tube is close at the end $x = 0$. Gas is confined in the region $L < x < 2L$ at pressure, density, and temperature p_1, ρ_1, T_0. Elsewhere the ambient state is p_0, ρ_0, T_0. The gas is a perfect gas. At $t = 0$ the barriers at $x = L, 2L$ are broken. If $p_1 = 1.1p_0$, and the approximation of acoustics is used, what is the pressure distribution in the tube at time $t = 7L/4c_0$? (Hint: The domain may be extended to the real line by making pressure even, velocity odd, in x.)

3. Establish uniqueness of the solution of the IVP for the wave equation in N dimensions using the energy method, in the following case:

(a) Show that if u is twice continuously differentiable and $u_{tt} - c^2\nabla^2 u = 0$, for $t > 0$, all $\mathbf{x}$, with $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_N^2}$, and also either u or u_t is identically zero outside a sphere of some large radius R centered at the origin, then the energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty}(u_t^2 + c^2(\nabla u)^2)dx_1\ldots dx_N$$

is constant in time. Use the fact that then the range of influence of the initial data lies inside the sphere of radius $R + ct$ at time t.

(b) Then obtain uniqueness by considering the null initial conditions $u_t = 0, u = 0$.

4. (a) Find the solution of Burger's wave equation $u_t + uu_x = 0$ with the initial conditions $u(x,0) = 0, x < 0; = x^2, 0 \le x \le 1; = 1, x > 1$.

5. Let $B(\eta)$ be a given function differentiable function, $B' \neq 0$. Show that if $u(x,t)$ is defined implicitly by $u = B(x - ut)$, so $u(x,0) = B(x)$, then $u(x,t)$ satisfies Burgers' equation $u_t + uu_x = 0$ for $t > 0$.

6. For the viscous Burgers equation derived in class, let $U = (\gamma+1)u/2$ and $\epsilon = 2\mu/3$= constant to get $U_t + UU_x - \epsilon U_{xx}$. The *Cole-Hopf* transformation is given by $U = -2\epsilon\frac{\psi_x}{\psi}$. Show that under this transformation, Burgers' equation reduces to

$$\frac{\partial}{\partial x}[\frac{\psi_t}{\psi} - \epsilon\frac{\psi_{xx}}{\psi}] = 0.$$

Hence show that a solution of Burgers' equation is generated by taking

$$\psi = 1 + t^{-1/2}e^{-\frac{x^2}{4\epsilon t}}.$$

Describe this solution by sketching $U\sqrt{t/\epsilon}$ as a function of $\eta = \frac{x}{2\sqrt{\epsilon t}}$ for various t.