Fluid Dynamics II PROBLEM SET 3 Due March 1, 2004

1. Solve the initial value problem

$$u_t + u^p u_x = 0, u(x,0) = 0, x < 0, = 1, x \geq 0.$$

Here p is an arbitrary positive integer. (Hint: In the region $0 < x/t < 1$ we have an expansion fan, $u = f(x/t)$.)

2. Consider the system

$$u_t + vu_x + u^2 v_x = 0, \;\; v_t + v^2 u_x + vv_x = 0$$

for $u(x,t), v(x,t)$. Find the characteristic velocities and the associated Riemann invariants as functions of u,v.

3. For the system of problem 2, solve the simple wave problem defined by the following conditions: $u(x,0) = 1, v(x,0) = 2$ for $x > 0$, $u(0,t) = v(0,t) = 0, t > 0$. Considering only the region $x \geq 0$, show that $x > 4t$ is a region of constant state, and fit an expansion fan in the region $0 < x < 4t$. Find an equation for the C_- characteristics in the expansion fan.

4. Verify the equations for the particle paths and the cross (C_-) characteristics for the expansion fan region of the piston receding at constant velocity ($u_p = U_0$), as discussed in class:

$$x = -(2c_0/(\gamma - 1))t + \frac{\gamma + 1}{\gamma - 1} c_0 t_0 (t/t_0)^Q$$

where $Q = \frac{2}{\gamma+1}$ for particle paths and $Q = \frac{3-\gamma}{1+\gamma}$ for the C_- characteristics.

5. Consider steady irrotational homentropic flow of a polytropic gas in three dimensions. Show that the equation for the velocity potential ϕ may be put into the form

$$c_s^2 \nabla^2 \phi = \nabla\phi \cdot (\nabla\phi \cdot \nabla\nabla\phi) + \frac{1}{2}(\gamma - 1)(\nabla\phi)^2 \nabla^2 \phi,$$

where c_s^2 is the stagnation sound speed (the speed of sound when $\vec{u} = 0$) squared.

6. Consider the equations for the unsteady one dimensional homentropic flow of a polytropic gas. Let a solution on $-\infty < x < +\infty$ be of the form $(u, \rho) = (U(x - Vt), R(x - Vt))$, where V is a constant and U and R are continuously differentiable functions. Show that necessarily u and ρ are constants. (Remark: If entropy is not constant and shocks occur, traveling wave solutions of this kind exist.)