Fluid Dynamics II PROBLEM SET 4 Due March 8, 2004

1. Show that a solution of $\beta^2\phi_{zz} - \phi_{rr} - r^{-1}\phi_r = 0$, $\beta = \sqrt{M^2-1}$ in the supersonic case, is given by

$$\phi = \int_0^{z-\beta r} \frac{f(\zeta)}{\sqrt{(z-\zeta)^2 - \beta^2 r^2}} d\zeta$$

where we assume $f(0) = 0$. Do this by setting $\zeta = z - \beta r \cosh\lambda$ to obtain

$$\phi = \int_0^{\cosh^{-1}(z/\beta r)} f(z - \beta r \cosh\lambda) d\lambda,$$

then carrying out the needed differentiations.

2. Consider unsteady one-dimensional flow of a polytropic gas. The *hodograph transformation* of gas dynamics takes x, t as dependent variables rather than u, ρ. Using the transformation of partials given by

$$\begin{pmatrix} u_x & u_t \\ \rho_x & \rho_t \end{pmatrix} = \begin{pmatrix} x_u & x_\rho \\ t_u & t_\rho \end{pmatrix}^{-1} = \frac{1}{J}\begin{pmatrix} t_\rho & -x_\rho \\ -t_u & x_u \end{pmatrix},$$

where $J = x_u t_\rho - x_\rho t_u$, show that the equations for u, ρ in homentropic flow are transformed into

$$Cw_\rho + Dw_u = 0,$$

where $w = (x,t)^T$ and C, D are matrix functions of u, ρ. These are the hodograph equations. From the eigenvalues of $C^{-1}D$ determine the charateristic curves for the hodograph equations. Hence show that the latter are curves of constant Riemann invariant $F_\pm$.

3.Consider the wave equation $u_t + (F(u))_x = 0$ with $F(u) = u(1-u)$. Describe the shock solution which results for the initial conditions $u = 1/4, x < 0, = 1/2, x > 0$. Give the shock speed, and sketch characteristics and shock position in the x-t plane.

4. Sketch the $x - t$ plane and determine explicitly the path of the shock wave for the following IVP:

$$u_t + uu_x = 0, \quad u(x,0) = \begin{cases} 1+x, & -1 \le x \le 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

5. Consider the Burgers viscous equation in the form

$$(\frac{\partial}{\partial t} + c_0\frac{\partial}{\partial x})u + \frac{\gamma+1}{2}u\frac{\partial u}{\partial x} - \frac{2\mu}{3}\frac{\partial^2 u}{\partial x^2} = 0.$$

Find a solution of the form $u = \frac{2}{\gamma+1}F(x-c_0t)$ representing a smooth transition from slightly subsonic velocity at $x = -\infty$ to slightly supersonic velocity at $x = +\infty$. Compute the viscous dissipation in the structure. (Note that the dissipation is independent of viscosity. This solution correctly describes the structure of a normal shock wave allowing for viscous stresses but neglecting thermal conductivity. Optional research problem: find the analogous solution including thermal conductivity.)