COMPLEX VARIABLES I. FINAL EXAM DEC. 19, 2006

ANSWERS

1 (a) $\frac{3-2i}{1-4i} = \frac{3-2i}{1-4i}\,\frac{1+4i}{1+4i} = \frac{3-2i+12i+8}{17} = \frac{11}{17} + i\,\frac{10}{17}$

(b) $(1+i)^{1/3} = \left(\sqrt{2}\,e^{i\pi/4}\right)^{1/3} = 2^{1/6}\,e^{i\frac{\pi}{12}}\left(1, e^{2\pi i/3}, e^{4\pi i/3}\right)$

$= 2^{1/6}\,e^{i\pi/12},\ 2^{1/6}\,e^{3\pi i/4},\ 2^{1/6}\,e^{17\pi i/12}$ $\quad 17\pi i/12$

$2^{1/6}\,e^{i\left(\frac{\pi}{12}+\frac{2k\pi}{3}\right)}$

(c) $\left|\frac{1}{1-z}\right| > 2 \Rightarrow |1-z| < 1/2$ the interior of the circle of radius 1/2 centered at $z=1$.

(d) $z=1$ is an essential singularity of $e^{\frac{1}{1-z}}$ $= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{(1-z)^n}$ and hence of $\frac{e^{1/1-z}}{(1+z^2)^2}$

$z=\pm i$ are poles of order 3 since $(1+z^2)^3 = (z+i)^3(z-i)^3$, and $e^{1/1-z}$ is analytic and $\neq 0$ at $z=\pm i$.

(e)(i) $\nabla^2(2xy)=0$ $\quad u_x = 2y = v_y \quad u_y = 2x = -v_x$

so $v = \frac{y^2}{2} - \frac{x^2}{2} + C.$

(ii) $\nabla^2 x^2 y = 2y \neq 0$ so f is not analytic.

2 By the Cauchy Integral formula.

$$f'(0) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z^2}\, dz$$

So.

$$|f'(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f|\, d\theta \leq \frac{2\pi}{2\pi} \cdot 1 = 1.$$

3 (a)
$$\frac{1}{(2z+1)(z-1)^2} = \frac{1}{(2(z-1)+3)(z-1)^2}$$

$$= \frac{1}{3} \frac{1}{\left[1+\frac{2}{3}(z-1)\right](z-1)^2}$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n (z-1)^{n-2}$$

(b) The series converges absolutely iff $|z-1| < \frac{3}{2} = R_{max}$

(c)
$$\frac{1}{(2z+1)(z-1)^2} = \frac{1}{(2(z-1)+3)(z-1)^2}$$

$$= \frac{1}{2(z-1)} \frac{1}{\left(1+\frac{3}{2(z-1)}\right)(z-1)^2}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n (-1)^n \frac{1}{(z-1)^{3+n}}$$

4 $\displaystyle\oint_{|z|=2} \frac{1+z^2}{(1-z)^3}$

$$= + \oint_{|z|=2} \frac{(1+z^2)}{(z-1)^3}$$

$$= 2\pi i \operatorname*{Res}_{z=1} \left(\frac{1+z^2}{(z-1)^3}\right)$$

$$= 2\pi i \left(\tfrac{1}{2}\cdot 2\right) = 2\pi i \qquad z = (z-1+1)$$

Or note $1+z^2 = 1 + (z-1)^2 + 2(z-1) + 1$

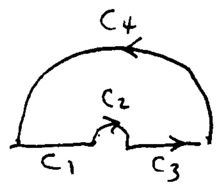
so $\operatorname*{Res}_{z=1} f = 1$

or use $\displaystyle\operatorname*{Res}_{z=1} = \frac{1}{2}(1+z^2)''\Big|_{z=1} = 1$

(5) Within $|z| = 2$ we see that we can choose $z^5 = f$ and $3z^2 - 1 = g$ so that $|f| = 2^5 = 32$, $|g| \leq 13$. Hence there are 5 roots in $|z| < 2$.

On $|z| = 1$, we may take $f = 3z^2$ $|f| = 3$, $g = z^5 - 1$, $|g| \leq 2$, to conclude that there are 2 roots in $|z| < 1$. Thus there are 3 roots in $1 < |z| < 2$, since there are no roots on $|z| = 1$.

(6) $$I = \int_0^{\infty} \frac{\sin x}{x(1+x^2)}\,dx = \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{+\infty} \frac{e^{ix}}{x(1+x^2)}\,dx$$



$$2I = \lim_{R\to\infty}\ \lim_{\varepsilon\to 0}\ \mathrm{Im}\int_{C_1+C_3} f(z)\,dz, \qquad f(z) = \frac{e^{iz}}{z(1+z^2)}$$

$$\int_{C_1+C_2+C_3+C_4} f(z)\,dz = 2\pi i\left[\operatorname*{Res}_{z=i} f\right] = 2\pi i\,\frac{e^{-1}}{i(2i)} = -i\pi e^{-1}$$

$\lim_{\varepsilon\to 0} \int_{-C_2} f(z)\,dz = +\pi i$ since $f(z)$ has a simple pole at $z = 0$ with residue 1.

$$\lim_{R\to\infty}\left|\int_{C_2} f(z)\,dz\right| \leq \frac{2}{R^2-1}\int_0^{\pi/2} e^{-\frac{\theta 2}{\pi}R}\,d\theta = \frac{\pi}{R(R^2-1)} \to 0 \text{ as } R\to\infty$$

Thus $2I = \mathrm{Im}\left(\pi i - \pi i e^{-1}\right) = \pi\left(1 - e^{-1}\right)$