Complex Variables I Final examination December 19, 1006

ANSWER ALL QUESTIONS. JUSTIFY YOUR ANSWERS.

1. (25 points):
(a) Find the real and imaginary parts of $\frac{3-2i}{1-4i}$.
(b) What are the three values of $(1+i)^{1/3}$, expressed in the form $r_k e^{i\theta_k}, k=1,2,3$?
(c) Sketch or describe the region of the z-plane determined by the inequality $\left|\frac{1}{1-z}\right| > 2$.
(d) Classify the singularities of $f(z) = \frac{e^{\frac{1}{1-z}}}{(1+z^2)^3}$.
(e) In the following, can f be an entire function of z? If so, what is $v(x,y)$?

$$(i)\ f = 2xy + iv(x,y), \quad (ii)\ f = xy^2 + iv(x,y)$$

2. (15 points) Prove the following: If $f(z)$ is analytic within and on the circle $|z|=1$ and $|f(z)| \leq 1$ on $|z|=1$, then $|f'(0)| \leq 1$.

3. (15 points) (a) Compute the Laurent expansion of $f(z) = \frac{1}{(2z+1)(z-1)^2}$ about $z=1$ that is valid in $|z-1| < R$, R a suffieicently small positive number; (b) What is the largest R such that the expansion in (a) represents the function in the open disc $|z-1| < R$? (c) Compute the Laurent expansion of $f(z)$ about $z=1$ valid in the region $|z-1| > R_{max}$, where R_{max} is the number found in (b).

4. Evaluate (15 points)

$$\oint_{|z|=2} \frac{1+z^2}{(1-z)^3} dz,$$

where the integral is taken once around the circle in the *clockwise* direction.

5. (15 points) Determine, using Rouché's theorem, the number of zeros (counting multiplicities) of $z^5 + 3z^2 - 1$ in the annulus $1 < |z| < 2$. (Hint: consider the regions $|z| < 2$ and $|z < 1$ separately.)

6. (15 points) Evaluate using residue theory

$$I = \int_0^\infty \frac{\sin x}{x(1+x^2)} dx.$$