Lecture Notes: Complex Variables II, Spring 2007

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1 Lecture 9: The Riemann mapping theorem

These notes will contain material you should know from the final lectures of the course, which will be drawn from sources other than the text.

The Riemann mapping theorem is a key theoretical result in complex analysis. The basic question is to determine the existence of conformal maps between two general domains D, D'/-. The simplest formulation is for two simply-connected domains. We can also simplify this to the existence of a map from the interior of a general simply-connected domain D onto the unit disk $|z| < 1$. A variant is the existence of a map for the exterior of a simply-connected domain D onto the exterior of the slit $|x| < 1, y = 0$. Since a conformal map of a conformal map is conformal, we can get from D to D' via one of these "canonical" domains, e.g. the interior of the unit disk.

1.1 The Schwarz lemma

It is of interest to ask, as a preliminary question, just what are the maps which take the unit disk onto itself, That is, what are the maps which carry the unit disk $|z| < 1$ onto itself in a one-one fashion. We have seen in an earlier lecture that there is a class of linear transformations which do this, namely

$$w = e^{i\beta} \frac{z - \alpha}{1 - \bar{\alpha} z}, \tag{1}$$

where β is an arbitrary real number and α is an arbitrary complex number. Thus there is at least a three-parameter family of such maps, for the case where D is also the unit disk. The Schwarz lemma will help us pin this class of maps down completely.

Lemma 1 *Let $f(z)$ be analytic and $|f(z)| \leq 1$ for $|z| < 1$. If $f(0) = 0$, then $|f(z)| \leq |z|$. Also ,if $|f(z_0)| = |z_0|$ for some z_0 in the disk, then $f(z) = kz$ for some k with $|k| = 1$.*

Note that this says that a bounded (by 1) analytic function which is known to have somewhere a value (here 0) smaller than the bound, then the bound on the function can be improved. The proof is quite simple. If $0 < r < 1$, it follows from the maximum principle and the assumed bound on $|f(z)|$ that

$$\Big|\frac{f(z)}{z}\Big| \leq \frac{1}{r}, \ |z| < r. \tag{2}$$

Since r can be made as close as we like to 1, we have

$$\Big|\frac{f(z)}{z}\Big| \leq 1, \ |z| < 1. \tag{3}$$

Thus $|f(z)| \leq |z|$ as stated. If equality holds at a point in $|z| < 1$, then by the maximum principle $|f(z)/z|$ is constant over the disk, yielding $f(z) = kz$.

As an application of the Schwarz lemma we will fully characterize all conformal maps which take the unit disk onto itself.

Lemma 2 *Any one-to-one conformal map of the unit disk onto itself is given by a linear map of the form* (1) *with* $|\alpha| < 1$.

To prove this let such a map be $f(z)$, and set

$$g(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}. \tag{4}$$

Now on the boundary $|f| = 1$ and therefore $|g| = 1$. By the maximum principle $|g| \leq 1$ in $|z| < 1$. Also $g(0) = 0$. Thus, by the Schwarz lemma, $|g(z) \leq |z|$. Since in fact $|g| = 1$ everywhere on the boundary, it follows that $g(z) = e^{i\beta}z$. Solving for f, we obtain (1) with $\alpha = -f(0)e^{-i\beta}$.

We can thus see that in the case where D=D'= unit disk, we have a three-parameter family of maps with parameters α (real and imaginary parts counted separately), and β. If we want to restrict the maps considered so that the answer is unique, we can impose three conditions. These are often determined by choosing a point ζ in the unit disk and requiring that $f(\zeta) = 0$ and $f'(\zeta)$ be real positive. This effectively sets $\alpha = \zeta$ and $\beta = 0$ (since the derivative at ζ is easily seen to be $e^{i\beta}/(1 - |\zeta|^2)$).

1.2 A perturbation problem

Before considering the Riemann mapping theorem for simply-connected domains, it is instructive to consider a related perturbation problem. Suppose D is the simply-connected domain whose boundary in polar coordinates is given by $r = R(\theta) = 1 - \epsilon g(\theta), 0 \leq \theta < 2\pi$. Here ϵ is a small number, and $0 < g < 1$. The idea is that to map this domain onto the unit disk, we only have to change it slightly by expanding outward a small distance along each radial line. Thus we would expect a mapping which does this to be close to the identity. We accordingly set

$$f(z) = z(1 + \epsilon A_1 + \epsilon^2 A_2 + \ldots), \ \ A_j = \sum_{n=0}^{\infty} a_{jn} z^n. \tag{5}$$

We now impose the condition that $|f| = 1$ on $r = R(\theta), 0 \leq \theta < 2\pi$. Thus we set

$$\left| e^{i\theta}(1 - \epsilon g(\theta))[1 + \epsilon A_1 + \epsilon^2 A_2 + \ldots]_{z=e^{i\theta}(1-\epsilon g(\theta))} \right| = 1. \tag{6}$$

To make the terms in ϵ cancel out, we see that we must have

$$g(\theta) = \Re \sum_{n=0}^{\infty} a_{1n} e^{in\theta}, a_{1n} = \alpha_n + i\beta_n. \tag{7}$$

The we see that $f(0) = 0$ and that, through terms of order ϵ, $f'(0) = 1 + \epsilon a_{10}$. If we want this positive, then $\beta_0 = 0$. We also see that α_0 determines the "expansion factor" to order ϵ. This suggests that the desired map will maximize $f'0)$. One could study this further by going to higher powers of ϵ.

1.3 The main theorem

Theorem 1 *(Riemann mapping theorem) If D, D' are both simply-connected domains possessing more then one boundary point, there always exists an analytic function which maps D conformally onto D' in a one-one manner.*

Remarks: In fact there exists always a 3-parameter family of such maps, but the results becomes unique if we require e.g. that $f(\zeta) = 0, f'(\zeta) > 0$, with $|\zeta| \in D$. Also to get from D to D' we can map D and D' onto the unit disk, since a composition of conformal maps is conformal. Finally, at least two boundary points are needed, since otherwise the boundary could be the point at infinity, and the desired mapping function would then satisfy $|f(z)| < 1$ at all finite points z, implying, by Liouville's theorem, that f =constant.

We can thus reformulate the mapping theorem as follows:

Theorem 2 *If D is a simply-connected domain with more than one boundary point, and ζ is a point in D, then there exists a unique function $w = f(z)$ which is analytic in D and maps D conformally onto the unit disk $|w| < 1$, with $f(\zeta) = 0, f'(\zeta) > 0$.*

We now give some but not all of the steps in a proof of Theorem 2 (Reference: Nehari's *Complex Analysis*, pp. 181-187. The idea is to obtain the desired mapping as the solution of an extremal problem wherein $f'(\zeta)$ is maximized. The steps are as follows:

(i) Introduce a set B of functions from which the desired map will be found.

(ii) Show that B is not empty.

(iii) Show that if $F \in B$ is not the mapping function (i.e. some point of the unit disk has no pre-image in D), then we can find a $G \in B$ such that $G'(\zeta) > F'(\zeta)$.

(iv) Construct a sequence of functions $F_n \in B$ such that $F'_{n+1}(\zeta) > F'_n(\zeta)$, which converges to the desired mapping function.

We shall consider the first three steps but not the fourth.

The set B will be the functions $F(z)$ which are analytic and one-one on D, and satisfy $|F(z)| < 1$ on D. Given a point ζ of D, the functions of B also satisfy $F(\zeta)=0, F'(\zeta)>0$. Evidently the desired mapping function belongs to B.

We first note

Lemma 3 *The desired mapping function $w=f(z)$ is distinguished by being an element of B with the largest derivative at ζ.*

To prove this consider the function $p(w)=F(f^{-1}(w))$, where F is any function of B and f is the desired mapping function. Since $f^{-1}(0)=\zeta$ and $F(\zeta)=0$, we see that $p(0)=0$. Also $|p(w)|<1$ since $|F(z)|<1$. By the Schwarz lemma $|p(w)/w|<1$ and therefore, letting $w\to 0$, $|p'(0)|<1$. But we see (cf. problem 1 of set 8) that

$$p'(0)=\frac{F'(\zeta)}{f'(\zeta)}, \tag{8}$$

and so $f'(\zeta)>F'(\zeta)$.

Now consider (ii). Let a,b be two boundary points of D, and set

$$u(z)=\sqrt{\frac{z-a}{z-b}}. \tag{9}$$

Here we take one of the two branches of the square root on D. Since $u^2(z_1)=u^2(z_2)$ implies $z_1=z_2$, we see that u is one-one. Also, if A is any complex number, the two values $\pm A$ cannot both be taken on D. Now if z_1 is a point of D and $w_1=u(z_1)$, there is a small neighborhood of z_1 lying in D mapping into $|w-w_1|<\gamma, \gamma>0$. Thus the values $w:|w+w_1|<\gamma$ cannot be assumeD by $u(z)$ on D. Thus

$$f_0(z)=\frac{\gamma}{u(z)+w_1} \tag{10}$$

satisfies $|f_0(z)|<1$ on D and is one-one there.

To satisfy the remaining conditions for B we set

$$f_1(z)=\frac{|f_0'(\zeta)|}{f_0'(\zeta)}\Big(\frac{f_0(z)-f_0(\zeta)}{1-\overline{f_0(\zeta)}f_0(z)}\Big). \tag{11}$$

Since the linear transformation $e^{i\beta}\big(\frac{w-\alpha}{1-\bar\alpha w}\big)$ maps the unit disk into itself, we see the $|f_1(z)|<1$ on D, $f_1(\zeta)=0$, and it is easy to check that $F'(\zeta)>0$. Thus $f_1\in B$ and so B is not empty.

Now consider (iii). Suppose that $F(z)\in B$ is not the desired mapping function. Then there is a point α, $|\alpha|<1$, such that $F(z)\neq\alpha$ on D. Set

$$\phi(z)=\sqrt{\frac{\alpha-F(z)}{1-\bar\alpha F(z)}}, \tag{12}$$

$$H(z)=\frac{\sqrt{\alpha}-\phi(z)}{1-\overline{\sqrt{\alpha}}\phi(z)}, \tag{13}$$

$$G(z) = \frac{|H'(\zeta)|}{H'(\zeta)} H(z). \tag{14}$$

From $|F(z)| < 1$ it follows that ϕ and G satisfy the same inequality, $|G(z)| < 1$. Also it can be shown (Problem 2 of set 8) that

$$G'(\zeta) = |H'(\zeta)| = \frac{1 + |\alpha|}{2\sqrt{|\alpha|}} F'(\zeta) > F'(\zeta). \tag{15}$$

Step (iv) is a bit involved but the idea should be clear, to keep finding alphas whose moduli converge to 1 and simultaneously constructing functions $G(z)$ whose derivative at ζ is monotone increasing. With a little work one can prove the existence of a sequence of functions converging to the desired unique mapping function.

2 Lecture 10: Asymptotic exapansions of integrals

2.1 Asymptotic expansions

In many practical problems exact solutions are difficult to obtain in closed form but approximate solutions can be computes without too much difficulty. The precise meaning of "approximate" needs to be clarified. For example a complex-valued function of z defined in some domain D may be studied near a point $z_0 \in D$. Suppose that $f(z_0) = 0$ and the zero is second order. The order symbol O is useful for expressing this point, i.e.

$$f(z) = O((z - z_0)^2), \quad z \to z_0. \tag{16}$$

If f is analytic at z_0 then $g(z) = f'(z_0)(z - z_0)$ clearly approximates f near z_0. We can express this using the "O"-symbol as

$$f(z) - g(z) = O((z - z_0)^2), \quad z \to z_0. \tag{17}$$

We can also make use of an order symbol which indicates"smaller order". This is the "o" symbol. To express the fact the RHS of the last expression is smaller than the size of g we can write

$$f(z) - g(z) = o(z - z_0), \quad z \to z_0. \tag{18}$$

We now define these symbols precisely. We say that $\phi(z) = O(\psi(z))$ as $z \to z_0$ in D provided that there exists a positive number A and neighborhood N of z_0 such that $|\phi(z)| \leq A|\psi(z)|$ provided $z \in N \cap D$.

Similarly we say that $\phi(z) = o(\psi(z))$ as $z \to z_0$ provided that, for any positive number ϵ there is a neighborhood N_ϵ of z_0 such that $|\phi(z)| \leq \epsilon|\psi(z)|$ provided $z \in N_\epsilon \cap D$. Note that this really is saying that $|\phi/\psi| \to 0$ as $z \to z_0$ in D.

Note that z might be a variable in a problem, or a parameter, real or complex. z_0 could be zero or the point at infinity. The restriction to a domain D is important. Often this is a sector of the z-plane, $\alpha \leq \arg(z) \leq \beta$. For example $e^z = o(z^n), z \to \infty,\ z \in S$, where $S : \pi/2 + \epsilon \leq \arg(z) \leq 3\pi/2 - \epsilon$ for any integer n and positive ϵ.

Often one has a function f of z and a real parameter ϵ, and an approximating function to $f(z, \epsilon)$ is sought valid as $\epsilon \to 0$. Often powers of ϵ are involved. Then an asymptotic series approximation $G_N(z, \epsilon)$ to f might have the form

$$G_N = g_0(z) + \epsilon g_1(z) + \ldots + \epsilon^N g_N(z), \tag{19}$$

with the property that

$$f(z, \epsilon) - g(z, \epsilon) = O(\epsilon^{N+1}) = o(\epsilon^N), \epsilon \to 0. \tag{20}$$

Note that we do not assume anything about the convergence of the sequence g_N as $N \to \infty$. I can certainly happen that there is no convergence, even though for each N the asymptotic property holds. This is a basic difference between convergent infinite series and asymptotic series whose partial sums form asymptotic approximations.

Example 1: Consider the integral

$$f(z) = \int_0^\infty \frac{e^{-t}}{1 + zt} dt. \tag{21}$$

We are interested in the behavior of this integral as $z \to 0$ in the domain $-\pi + \epsilon \leq \arg z < \pi - \epsilon$. One method useful for some integrals of this type involves repeated integration by parts. We see that

$$f(z) = -\frac{e^{-t}}{1 + zt}\Big|_0^\infty - z \int_0^\infty \frac{e^{-t}}{(1 + zt)^2} dt, \tag{22}$$

$$= 1 - z + 2z^2 \int_0^\infty \frac{e^{-t}}{(1 + zt)^3} dt = \ldots \tag{23}$$

$$= \sum_{n=0}^{m} (-1)^n n! z^n + (-1)^{m+1}(m + 1)! z^{m+1} \int_0^\infty \frac{e^{-t}}{(1 + zt)^{m+2}} dt. \tag{24}$$

By writing $z = re^{i\theta}$ an determining the maximum value of $|1 + zt|^{-1}$ we have

$$\Big| \int_0^\infty \frac{e^{-t}}{(1 + zt)^{m+2}} dt. \Big| < (\sin \epsilon)^{-(m+2)}, \tag{25}$$

which is $O(1)$ as $z \to 0$. Thus the series $g_m = \sum_{n=0}^m (-1)^n n! z^n$ is asymptotic to $f(z)$ as $z \to 0$ with an error $O(z^{m+1})$. Note that g_m is horibly divergent as $m \to \infty$.

Example 2: We study

$$I(k) = \int_k^\infty \frac{e^{-t}}{t} dt \tag{26}$$

in the limit $k \to +\infty$. After integrating by parts N times we obtain

$$I(k) = e{-}k\Big[\frac{1}{k} - \frac{1}{k^2} + \frac{2!}{k^3} - \ldots + \frac{(-1)^{N-1}(N-1)!}{k^N}\Big] + (-1)^N N! \int_k^\infty \frac{e^{-t}}{t^{N+1}} dt. \tag{27}$$

Since

$$\Big|(-1)^N N! \int_k^\infty \frac{e^{-t}}{t^{N+1}} dt\Big| \le e^{-k}/k^{N+1} \tag{28}$$

we see that we have an asmptotic expansion of $I(k)$. Again the infinite series being developed here is not convergent for any finite k.

2.2 The method of stationary phase

We now come to an interesting and useful example of asymptotic approximation to a complex-valued function represented by an integral. Examples of this kind arise frequently in the theory of wave propagation. Consider the partial-differential equation i real x, t representing one-dimensional space and time.

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^3} = 0. \tag{29}$$

One way to find solutions is to make use of complex exponentials and substitute a function of the form

$$u = f(\lambda) e^{i(\omega t + \lambda x)}. \tag{30}$$

Here λ and ω are real parameters. Physically $\frac{\omega}{2\pi}$ is a frequency and $2\pi/\lambda$ is a wavelength. Substituing into (29) we see that $fi(\omega + \lambda^3) = 0$, which implies that $\omega = -\lambda^3$. We can thus obtain a large class of solutions by continuously combining the solutions of various λ. In particular for smooth λ we can obtain a solution as an integral,

$$u = \int_a^b f(\lambda) e^{i(\lambda x - \lambda^3 t)} d\lambda. \tag{31}$$

The question we want to answer here concerns the behavior of (31) as $t \to \infty$ with x/t fixed. Physically, the question is to determine the wave structure seen at large times by an observer moving with velocity x/t. We can think of the general case as an integral with the structure

$$u = \int_a^b f(\lambda) e^{it\phi(\lambda)} d\lambda,\ t \to \infty. \tag{32}$$

Consider first the special case where $\phi = \lambda^2, f = 1, a = -1, b = 1$. We then have the integral

$$\int_{-1}^{1} e^{it\lambda^2} d\lambda = \int_{-\infty}^{+\infty} e^{it\lambda^2} d\lambda - I_+ - I_-, \tag{33}$$

where

$$I_+ = \int_1^\infty e^{it\lambda^2} d\lambda . I_- = \int_{-\infty}^{-1} e^{it\lambda^2} d\lambda. \tag{34}$$

Now

$$\int_{-\infty}^{+\infty} e^{it\lambda^2} d\lambda = \frac{1}{\sqrt{t}} \int_{-\infty}^{+\infty} e^{ix^2} dx = \frac{1}{\sqrt{t}} \sqrt{\pi} e^{i\pi/4}. \tag{35}$$

(Recall that we can compute $\int_0^\infty (\cos(x^2), \sin(x^2)) dx$ by integrating e^{iz^2} around the contour going from origin to $(R, 0)$, then along the circle $|z| = R$ to $Re^{i\pi/4}$, then to the origin along the ray $\arg z = \pi/4$, see page 266 of text.)

We will now show that the contribution just computed dominates over the contributions from $I_\pm$. For I_+ we note that, by integration by parts

$$I_+ = -\frac{e^{it}}{2it} + \frac{1}{2it} \int_1^\infty \lambda^{-2} e^{i\lambda^2 t} d\lambda. \tag{36}$$

It follows that $|I_+| \le \frac{1}{2t}(1 + \int_1^\infty \lambda^{-2} d\lambda = 1/t$. The same estimate holds for I_-. Thus

$$|I_\pm| = O(t^{-1}), t \to \infty. \tag{37}$$

The dominant contribution, of order $1/\sqrt{t}$ comes from the integral whose domain covers the point $\lambda = 0$ where λ^2 has its derivative equal to zero. In other words, as λ varies under the integral, the function $\phi(\lambda)$ is steadily changing as long as ϕ' does not vanish. Therefore as long as this is the case the function $e^{it\phi(\lambda)}$ will oscillate, and since t is large it will oscillate rapidly. The effect is to cause significant cancelation from the positive and negative contributions to the integral, over the domain of integration where $\phi' \neq 0$. As for the effect of $f(\lambda)$, the number of continuous derivatives that it has will determine the extent of the integrations by parts that can be performed on the interval where ϕ' does not vanish. When these contributions remain sub-dominant the main effect of f on the dominant contribution comes from the behavior of f at the point of vanishing of ϕ'.

At a point where ϕ' vanishes we say that the *phase* of $e^{it\phi}$ is *stationary*, hence the name of this method.

We now calculate formally the dominant contribution to (32) as $t \to \infty$, based on the above examples and ideas. We assume that ϕ' has a unique zero $\lambda = c$ in (a, b), and that $\phi''(c) \neq 0$. We will use the symbol $\sim$ to indicated the leading term of an asymptotic approximation. We then claim that under suitable conditions on f we will have

$$\int_a^b f(\lambda) e^{it\phi(\lambda)} d\lambda \sim f(c) e^{it\phi(c)} \int_{-\infty}^{+\infty} e^{it\phi''(c)(\lambda - c)^2} d\lambda,\ t \to \infty. \tag{38}$$

We thus obtain

$$\int_a^b f(\lambda) e^{it\phi(\lambda)} d\lambda \sim f(c) e^{it\phi(c)} \frac{\sqrt{\pi} e^{\frac{i\pi}{4} \operatorname{sgn}(\phi''(c))}}{\sqrt{|\phi''(c)| t}},\ t \to \infty. \tag{39}$$

In particular, this result hold if f, ϕ are infinitely differentiable on (a, b), f vanishes to all orders at a, b, and $f(c) \neq 0$, the error then being $O(1/t)$. Clearly there are many special cases that might be consider, where $\phi''(c) = 0$, where $f(c) = 0$, and where more than one stationary point exists in the interval of integration. (Reference: Ablowitz and Fokas, chapter 6, especially section 6.3.3. Anyone interested in a elegant short book on asymptotic expansions should get a copy of A. Erdélyi's book, *Asymptotic Expansions*, published by Dover.)

2.3 Asymptotic expansion of the Bessel function $J_n(r)$ for large r.

The Bessel function $J_n(r)$ can be defined by by the integral

$$J_n(r) = \frac{i^{-n}}{\pi} \int_0^{\pi} e^{ir\cos\theta} \cos n\theta \, d\theta = \frac{i^{-n}}{2\pi} \int_0^{2\pi} e^{ir\cos\theta} \cos n\theta \, d\theta. \tag{40}$$

A standard result is that for large positive r

$$J_n(r) = \sqrt{\frac{2}{\pi r}} \cos(r - \frac{1}{2}n\pi - \frac{1}{4}\pi) + O(r^{-1}), \; r \to +\infty. \tag{41}$$

Let us see how this result can be obtained by the method of stationary phase. Since the derivative of $\cos\theta$ vanishes at $\theta = 0, \pi$ there will be two contributions from the two stationary points. We thus get

$$J_n(r) \sim \frac{i^{-n}}{2\pi} \Big[e^{ir} \int_{-\infty}^{+\infty} e^{-ir\theta^2/2} \, d\theta + \cos n\pi e^{-ir} \int_{-\infty}^{+\infty} e^{ir\theta^2/2} \, d\theta \Big], \tag{42}$$

$$\sim \frac{1}{\sqrt{r}} \frac{\sqrt{2\pi}}{2\pi} e^{-in\pi/2} \Big[e^{ir - i\pi/4} + e^{in\pi} e^{-ir + i\pi/4} \Big] \tag{43}$$

thus yielding (41).

3 Lecture 11: The method of steepest descent

We have seen that the equation (29) can be solving by a superposition of dispersive waves

$$u = \int_a^b f(\lambda) e^{i(\lambda x - \lambda^3 t)} d\lambda \sim f(c) e^{it\phi(c)} \frac{\sqrt{\pi} e^{\frac{i\pi}{4}\mathrm{sgn}(\phi''(c))}}{\sqrt{|\phi''(c)| t}}, \; t \to \infty + O(1/t), t \to \infty, \tag{44}$$

where $\phi(\lambda) = \lambda \frac{x}{t} - \lambda^3, c = \sqrt{x/(3t)}$ and we assume c but not $-c$ is contained in (a, b). We thus have

$$u \sim \frac{f(c)\sqrt{\pi} e^{i\pi/4}}{\sqrt{2\sqrt{3}} (x/t)^{1/2} \sqrt{t}} e^{it \frac{2}{3\sqrt{3}} (x/t)^{3/2}} = \frac{1}{t^{1/3}} F(\eta), \; \eta = x/(3t)^{1/3}. \tag{45}$$

If we substitute $u = \frac{1}{t^{1/3}}F(\eta)$ into (29), we obtain

$$F + \eta F' + F''' = (\eta F + F'')' = 0. \tag{46}$$

Thus we are able to satisfy (29) by *any* function of the form $u = \frac{1}{t^{1/3}}F(\eta)$ by solving the ordinary differential equation

$$F_{\eta\eta} + \eta F = 0. \tag{47}$$

Setting $z = -\eta, , F(\eta) = w(z)$ in (47) gives us the standard form of *Airy's differential equation*:

$$w_{zz} - zw = 0. \tag{48}$$

The point of this is to move to a class of approximations to integrals obtained by a kind of generalization of the method of stationary phase. Recall that method needed to use the calculation of the Fresnel integrals as we did earlier using contour methods, an approach which in essence replaces rapid oscillations of the kernel by its exponential decay (we compute the integral of $\cos(tx^2)$ by converting it to an integral of e^{-tx^2}.) The *method of steepest descent* in effect does all of this in one step, by considering simultaneously the real and imaginary parts of a complex exponential.

We will now illustrate this method by finding solutions of Airy's equations and then approximating them asymptotically for large $|z|$.

3.1 Representation of solutions of ODEs by contour integrals

We describe now a very useful techniques for finding and studying solutions of certain linear ordinary differential equations, using Airy's to introduce the ideas. We try to solve (48) in the form

$$w = \int_C e^{z\sigma} v(\sigma) d\sigma \tag{49}$$

where σ is a complex variable and C is a certain contour of integration which is to be identified as part of the construction of the solution. Writing Airy's equation as $Lw = 0$, then we have from (49) that

$$Lw = \int_C v(\sigma) L(e^{z\sigma}) d\sigma = \int_C v(\sigma)(\sigma^2 - z) e^{z\sigma} d\sigma = 0. \tag{50}$$

We now note that integration by parts gives

$$\int_A^B v(\sigma) z e^{z\sigma} d\sigma = (v(\sigma)e^{z\sigma})\big|_A^B - \int_A^B v'(\sigma) z^{z\sigma} d\sigma. \tag{51}$$

Thus

$$\int_C v(\sigma) L(e^{z\sigma}) d\sigma = -(v(\sigma)e^{z\sigma})\big|_A^B + \int_A^B (v\sigma^2 + v') e^{z\sigma} d\sigma. \tag{52}$$

Therefore we have a solution of (48) under two conditions:

(i) The contour extends from point A to point B such that the integral exists and $(v(\sigma)e^{z\sigma})\big|_A^B = 0$. (Note that A or B or both could be the point at infinity.)

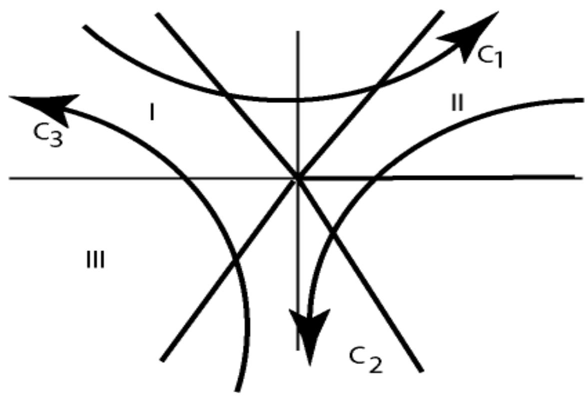
(ii) $v\sigma^2 + v' = 0$ or

$$v = Ce^{-\sigma^3/3}. \tag{53}$$

We take $C = \frac{1}{2\pi i}$, set $\sigma = iz^{1/2}\zeta$, and define $x = z^{3/2} > 0$. The endpoints A, B will be taken at infinity, as determined below. The Airy function we shall define is

$$Ai(x^{2/3}) = \frac{x^{1/3}}{2\pi}\int_C e^{xh(\zeta)}d\zeta,\ h = i(\zeta + \frac{1}{3}\zeta^3). \tag{54}$$

The contour C will be defined such that as $\zeta \to \infty$ $e^{i\zeta^3} \to 0$. Thus C will begin at ∞ in the sector $2\pi/3 < \arg\zeta < \pi$ and end at ∞ in the sector $0 < \arg\zeta < \pi/3$, see figure 1 below.



Note that the sectors I, II, III, each being $\pi/3$, are the sectors where $e^{i\zeta^3} \to 0$ as infinity is approached. Thus the contours $C_k, k = 1, 2, 3$ as shown in the figure will yield three solutions of Airy's equation. Since $e^{xh(\zeta)}$ is an entire function of ζ,

$$\int_{C_1+C_2+C_3} e^{xh(\zeta)}d\zeta = 0 \tag{55}$$

by Cauchy's theorem, implying that these three solutions are linearly dependent. A second solution of the equation can be defined, f0r example, as

$$Bi(x^{2/3}) = \frac{x^{1/3}}{2\pi}\int_{C_2-C_3} e^{xh(\zeta)}d\zeta,\ h = \zeta + \frac{1}{3}\zeta^3. \tag{56}$$

We focus here on A_i.

Recalling that we are taking $x > 0$, we seek an asymptotic description of A_i as $x \to \infty$. The idea now is to deform C to pass through a point where $h'(\zeta) = 0$. At such a point, ζ_0 say, we will have

$$h = h(\zeta_0) + h''(\zeta_0)\frac{1}{2}(\zeta - \zeta_0)^2 + \ldots, \tag{57}$$

If $\zeta = \xi + i\eta$, the real and imaginary pars of h will be harmonic functions of ξ, η, and will have not local maxima at finite ζ, so ζ_0 will be a saddle point. The idea is then to deform the contour to go over the col of the saddle along a path which has a local maximum and is the steepest path down from the maximum. Since Re(h)= constant and Im(h)= constant will always intersect at right angles, the steepest path will be a line of constant Im(h).

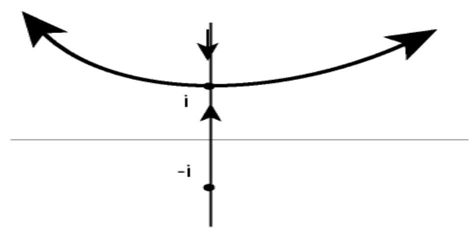
Now $h'(\zeta = i(1+\zeta^2) = 0$ when $\zeta = \pm i$. We thus take $\zeta_0 = i$.

$$\mathrm{Im} h(\zeta) = \frac{1}{3}\xi^3 - \xi\eta^2 + \xi, \tag{58}$$

so that Im $h(i)$=0. Since the steepest path down will go through the col $\zeta_0 = i$, the steepest path must satisfy

$$\xi(\xi^2 - 3\eta^2 + 3) = 0. \tag{59}$$

Now Re $h = \frac{1}{3}\eta^3 - \xi\eta - \eta$, and so Re $h(i)$=-2/3. Also the path $\xi = 0$ from (59) *increases* Re h as we move along it away from the col, so the steepest path *down* is along the hyperbola $\xi^2 - 3\eta^2 + 3) = 0$. We show the situation in figure 2, where the arrows indicate the direction where Re h is decreasing.



We now take advantage of the steepest path and first consider the integral from i to $i + \infty$ along the line $\eta = 1$. Call this I_1. Note that

$$h(\zeta) - h(i) = -(\zeta - i)^2 + \frac{i}{3}(\zeta - i)^3. \tag{60}$$

Let $\zeta - i = \sqrt{u}, u > 0$ in I_1. Then

$$I_1 = \frac{x^{1/3}}{2\pi} e^{-\frac{2}{3}x} \int_0^\infty e^{-xu} e^{\frac{1}{3}ixu^{3/2}} \frac{1}{2\sqrt{u}} du. \tag{61}$$

What we have gained here is the exponential decay of the integrand for $x > 0$. The integral clearly exists. To obtain an asymptotic series for large positive x

we expand the second exponential and integrate term by term, to obtain

$$I_1 \sim \frac{x^{-1/6}}{4\pi} e^{-\frac{2}{3}x} \sum_{n=0}^{\infty} \frac{i^n \Gamma(\frac{3}{2}n + \frac{1}{2})}{3^n \; n! \; x^{n/2}}. \tag{62}$$

Here recall the definition of the gamma function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \tag{63}$$

The integral from $-\infty$ to i along $\eta = 1$, call it I_2, is readily seen to be the complex conjugate of I_1. Putting these together and simplifying we obtain, finally,

$$Ai(x^{2/3}) \sim \frac{x^{-1/6}}{2\pi} e^{-\frac{2}{3}x} \sum_{m=0}^{\infty} \frac{\Gamma(3m + 1/2)}{(2m)!} (-9x)^{-m}. \tag{64}$$

If we now set $x^{2/3} = z$ we have

$$Ai(z) \sim \frac{1}{2\pi z^{1/4}} e^{-\frac{2}{3}z^{3/2}} \sum_{m=0}^{\infty} \frac{\Gamma(3m + 1/2)}{(2m)!} (-9z^{3/2})^{-m}. \tag{65}$$

This result holds for complex z as long as Re $z^{3/2} > 0$, e.g. for $|\arg z| < \pi/3$.

3.2 Some related integrals

We consider

$$f(x) = \int_0^{\infty} e^{xh(\zeta)} d\zeta, \; h = i(\frac{1}{3}\zeta^3 + \zeta). \tag{66}$$

We first let $x > 0$ and consider the path to be along the positive ξ axis. Since we are dealing with the same h as before, we know the steepest path from the col at i, so we may deform the path to take advantage of that result and write

$$\int_0^{\infty} = \int_0^{i} + \int_i^{i+\infty} = I_0 + I_1. \tag{67}$$

For I_1 we now have our previous result minus some prefactors:

$$I_1 \sim x^{-1/2} e^{-\frac{2}{3}x} \sum_{n=0}^{\infty} \frac{i^n \Gamma(\frac{3}{2}n + \frac{1}{2})}{3^n \; n! \; x^{n/2}}. \tag{68}$$

For I_0 we set $\zeta = iu$ and obtain

$$I_0 = i \int_0^1 e^{-xu} e^{-\frac{1}{3}xu^3} du \sim i \sum_{n=0}^{\infty} \frac{1}{3^n \; n! x^{2n+1}} \int_0^x e^{-t} t^{3n} dt. \tag{69}$$

We now show that

$$\int_0^x e^{-t} t^{3n} dt = (3n)! + o(1), \; x \to \infty. \tag{70}$$

Integrating by parts

$$\int_x^\infty e^{-t} t^{3n} dt = e^{-x} x^{3n} + 3n \int_0^x e^{-t} t^{3n-1} dt. \tag{71}$$

Continuing in this way, we obtain

$$\int_x^\infty e^{-t} t^{3n} dt = e^{-x} P(x), \tag{72}$$

where $P(x)$ is a polynomial of degree $3n$. Thus, since $e^{-x}x^N = o(1)$ as $x \to \infty$ for any positive integer N, we see that

$$\int_0^x e^{-t} t^{3n} dt = \int_0^\infty e^{-t} t^{3n} dt + o(1) = (3n)! + o(1),\ x \to \infty. \tag{73}$$

Noting that I_1 is exponentially small relative to I_0, we have the asymptotic expansion of $f(x)$ in the form

$$f(x) \sim i \sum_{n=0}^{\infty} \frac{(3n)!}{3^n\ n! x^{2n+1}}. \tag{74}$$

Consider now the integral

$$f(x) = \int_0^1 e^{ix\zeta^3} d\zeta,\ x > 0 \tag{75}$$

With $h(\zeta) = i\zeta^3$, h has a zero of order 2 at $\zeta = 0$. We will try to get from 0 to 1 by taking steepest paths to infinity from both $\zeta = 0$ and $\zeta = 1$. We thus write

$$\int_0^1 = \int_{C_1} + \int_{C_2} = I_1 - I_2. \tag{76}$$

For I_1 we look at the steepest path from $\zeta = 0$, where Im $i\zeta^3$=0. This occurs along the rays $\arg \zeta = \pm\pi/6, \pm\pi/2, \pm 5\pi/6$. By checking Re $i\zeta^3$ on these paths we see that the only reducing $|e^{i\zeta^3}|$ as we move from 0 are $\pi/6, 5\pi/6, 3\pi/2$. We choose the ray $\pi/6$ and set $\zeta = e^{i\pi/6}u$. Thus

$$I_1 = e^{i\pi/6} \int_0^\infty e^{-xu^3} du = e^{i\pi/6}\Gamma(1/3)\frac{1}{3x^{1/3}}. \tag{77}$$

For the steepest path our of $\zeta = 1$we look at Im h=1 or $\xi^3 - 3\xi\eta^2 = 1$. Since Re h= $\eta^3 - 3\eta\xi^2$ along this path, $|e^{i\zeta^3}|$ decreases maximally from $\zeta = 1$ moving along the line $1 + iu, u > 0$. If we set $\zeta^3 = 1 + iu$ we obtain

$$I_2 = \frac{i}{3} \int_0^\infty e^{-xu+ix}(1 + iu)^{-2/3} du. \tag{78}$$

Using the binomial theorem we have

$$(1+iu)^{-2/3}du = \sum_{n=0}^{\infty} \frac{\Gamma(1/3)}{\Gamma(n+1)\Gamma(1/3-n)}(iu)^n. \tag{79}$$

A useful identity for the gamma function is

$$\Gamma(z)\Gamma(1-z) = -z\Gamma(-z)\Gamma(z) = \frac{\pi}{\sin \pi z}. \tag{80}$$

Using this identity after inserting (79) into (78), we find

$$I_2 \sim e^{ix} \sum_{n=0}^{\infty} (ix)^{-n-1} \frac{\Gamma(n+2/3)}{\Gamma(-1/3)}. \tag{81}$$

Thus

$$f(x) \sim e^{i\pi/6}\Gamma(1/3)\frac{1}{3x^{1/3}} - e^{ix} \sum_{n=0}^{\infty} (ix)^{-n-1} \frac{\Gamma(n+2/3)}{\Gamma(-1/3)},\ x \to \infty. \tag{82}$$

3.3 The Stokes phenomenon

If $f(z)$ is analytic at infinity then $g(z) = f(1/z)$ is analytic at the origin and represented by it's Taylor series there, which also serves as an asymptotic expansion as $z \to 0$. If, on the other hand, $f(z)$ is not analytic at infinity then its asymptotic expansion as $z \to \infty$ can change abruptly as $\arg z$ varies. The phenomenon, associated with the name of Stokes, will be illustrated by considering the expansion of

$$I(z) = \int_0^{\infty} \frac{e^{-zt}}{1+t^4} dt \tag{83}$$

as $z \to \infty$. Assume first that $t > 0$ and $\text{Re } z > 0$. Then by expanding $(1+t^4)^{-1}$ we see that

$$I(z) \sim \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(4n+1)}{z^{4n+1}} = \frac{1}{z} - \frac{4!}{z^5} + \dots. \tag{84}$$

This expansion is in fact valid whenever the integrals converge, and this is true for $|\arg z| < \pi/2$.

Let us now deform the path of integration to be along the negative $\text{Im}(t)$-axis, from 0 to $-i\infty$. In so deforming the contour, we must take into account the simple pole at $t = e^{-i\pi/4}$. We then find that

$$I(z) = \int_0^{-i\infty} \frac{e^{-zt}}{1+t^4} dt + 2\pi i \frac{e^{-ze^{-i\pi/4}}}{4e^{-3\pi i/4}}. \tag{85}$$

Now the integral on the right converges when iz has negative real part, or when $0 < \arg z < \pi$. Since this region overlaps the region $|\arg z| < \pi/2$, the same function will be represented by either integral.

But note that the pole contribution has the property that it is exponentially small for $0 < \arg z < 3\pi/4$, but is exponentially *large* for $3\pi/4 < \arg z < \pi$. Thus in this last region the pole contribution dominates the infinite series in the asymptotic expansion. Thus the nature of the expansion changes abruptly on the *Stokes line* $\arg z = 3\pi/4$. A similar calculation involving deformation of the contour to the positive Im t-axis shows that the ray $\arg z = -3\pi/4$ is also a Stokes line, with an exponentially large residue from the pole at $t = e^{i\pi/4}$. Since these exponentially large terms are different in the sectors $3\pi/4 < \arg z < \pi$ and $-\pi < \arg z < -3\pi/4$ it follows that $\arg z = \pi$ is also a Stokes line.

References for this section: Erdelyi pp. 36-46, Ablowitz and Fokas pp. 298-300, pp. 488-490 and section 6.4.

4 Lecture 12: The gamma function

Recall that the gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{86}$$

Here we take x=Re $(z) > 0$ to insure convergence of the integral at $t = 0$, since $|t^{z-1}| = t^{x-1}$. Now

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt = -t^z e^{-t}|_0^\infty + z\Gamma(z) = z\Gamma(z). \tag{87}$$

If n is a positive integer, then induction on (87) gives,

$$\Gamma(n+1) = n!. \tag{88}$$

One sometimes writes $\Gamma(1+z) = z!$ for complex z, to emphasize the fact that (86) is a generalization of the factorial function. In fact the term "factorial function" also applies to $\Gamma(z+1)$.

If Re$(z) > 0$, $\Gamma(z)$ is an analytic function of z. Indeed

$$\frac{1}{h}[\Gamma(z+h) - \Gamma(z)] = \frac{1}{h}\int_0^\infty (t^h - 1)t^{z-1}e^{-t}dt \to \int_0^\infty t^{z-1}(\ln t)e^{-t}dt \tag{89}$$

as $h \to 0$.

It is easy to define $\Gamma(z)$ beyond the domain Re $(z) > 0$. For example, from (87) we use may the fact that $\Gamma(z+1)$ is defined by for Re $(z+1) > 0$, or Re $(z) > -1$, to find that $z\Gamma(z)$ is also defined and analytic there. Continuing in this way $\Gamma(z+2) = z(z+1)\Gamma(z)$ is analytic in Re $(z) > -2$, and

$$z(z+1)(z+2)\cdots(z+n-1)\Gamma(z) = \Gamma(z+n) \tag{90}$$

is analytic in Re $(z) > -n$. In this domain

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)\cdots(z+n-1)}, \tag{91}$$

showing that $\Gamma(z)$ is analytic at all finite points of the complex plane with the exception of simple poles at $0, -1, -2, \ldots$. A few special values of Γ are $\Gamma(1) = 0! = 1$, $\Gamma(1/2) = \sqrt{\pi}$ and

$$\Big(\frac{1}{2}\Big)! = \frac{1}{2}\sqrt{\pi}, \Big(\frac{3}{2}\Big)! = \frac{3}{2}\frac{1}{2}\sqrt{\pi}, \Big(-\frac{3}{2}\Big)! = -2\sqrt{\pi}, \Big(-\frac{5}{2}\Big)! = \frac{4}{3}\sqrt{\pi}. \tag{92}$$

(See problem (1) of set 10.)

Analytic continuation of $\Gamma(z)$ can also be accomplished by contour deformation We introduce the contour integral

$$\int_C t^{z-1}e^{-t}dt \tag{93}$$

where C is taken to surround the origin and begin and end at $z = +\infty$, see figure 3(a). The only possible finite singularity of the integrand is at $t = 0$, so we may deform the contour as shown in figure 3(b), the circle denoting the circle of radius R. Now the two integrals on either side of the positive $\mathrm{Re}(t)$-axis add up to

$$I_R = (1 - e^{2\pi i z})\int_R^\infty t^{z-1}e^{-t}dt. \tag{94}$$

The integral around the circle, I_0 say, satisfies

$$|I_0| \leq 2\pi e R^{\mathrm{Re}(z)}. \tag{95}$$

Thus if $\mathrm{Re}(z) > 0$, I_0 tends to 0 as $R \to 0$. Taking the limit, we obtain

$$\int_C t^{z-1}e^{-t}dt = (1 - e^{2\pi i z})\int_0^\infty t^{z-1}e^{-t}dt, \tag{96}$$

or

$$\Gamma(z) = \frac{1}{(1 - e^{2\pi i z})}\int_C t^{z-1}e^{-t}dt. \tag{97}$$

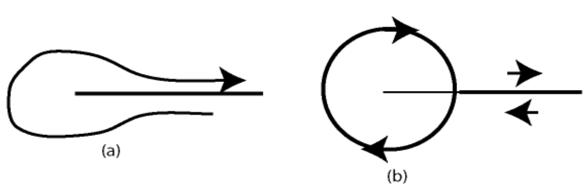


Figure 3

Now we observe that the RHS of (97) involves a contour integral which is an analytic function of z for all finite values of z. Thus (97) provides the analytic continuation of $\Gamma(z)$ to all finite values of z except possibly for the zeros of $1 - e^{2\pi i z}$. We know that the zeros $0, -1, -2, \ldots$ are indeed poles of $\Gamma(z)$. It can be shown that the zeroes $1, 2, \ldots$ of this function are in fact points of analyticity of the RHS of (97) (problem 2(b) of homework 10). Thus we have the desired analytic continuation to all points except the poles at $0, -1, -2, \ldots$.

4.1 The reflection identity

We now establish the useful and beautiful identity

$$z!(-z)! = \frac{\pi z}{\sin \pi z}. \tag{98}$$

This can also be written

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{99}$$

To prove it, we have

$$u!v! = \int_0^\infty e^{-x} x^u dx \int_0^\infty e^{-y} y^v dy = \lim_{M\to\infty} \int_0^M \int_0^M e^{-(x+y)} x^u y^v dxdy. \tag{100}$$

We now integrate over the triangle (1) of figure 4 by setting $z = x + y$ and eliminating y. We are then going to integrate by "strips" along the diagonals $z =$ constant, with z then varying from 0 to M. Thus we get

$$\int_0^M \int_0^z e^{-z} x^u (z-x)^v dxdz. \tag{101}$$

Putting $x = tz$ we then have

$$\int_0^M \int_0^1 z^{u+v+1} e^{-z} t^u (1-t)^v dtdz = \int_0^1 t^u (1-t)^v dt \int_0^M z^{u+v+1} e^{-z} dz. \tag{102}$$

As $M \to \infty$ in tis we get

$$(u+v+1)! \int_0^1 t^u (1-t)^v dt. \tag{103}$$

We next show that the integral over the triangle (2) of figure 4 tends to zero at $M \to \infty$. Indeed , the integrand there satisfies

$$|e^{-(x+y)} x^u y^v| \le e^{-M} M^{(\mathrm{Re}(u+v))}, \tag{104}$$

so the integral is bounded by $\frac{1}{2} e^{-M} M^{(\mathrm{Re}(u+v)+2)} \to 0$ as $M \to \infty$.

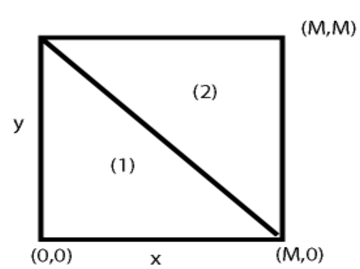


Figure 4

Thus, for Re$(u) > -1$ and Re$(v) > -1$ we have the important relation

$$\int_0^1 t^u(1-t)^v dt = \frac{u!v!}{(u+v+1)!}. \tag{105}$$

Setting $t = \frac{s}{1+s}$ in (105) we obtain

$$\int_0^\infty \frac{s^u}{(1+s)^{u+v+2}} ds = \frac{u!v!}{(u+v+1)!}. \tag{106}$$

We now revert to the use of z we began with, by setting $u = z, v = -z$, we see that the integral on the left of (106) exists if $|\text{Re}z| < 1$ and we have

$$\int_0^\infty \frac{s^z}{(1+s)^2} ds = \frac{\pi z}{\sin \pi z}, \tag{107}$$

see problem 3 of homework 10. Thus

$$z!(-z)! = \frac{\pi z}{\sin \pi z}. \tag{108}$$

This result now extends to all z other than the positive and negative integers by analytic continuation. I find the formula (108) easier to remember than the form (99), and there is the useful check at $z = 0$.

4.2 Gauss' definition of $z!$

Let us, in (105), replace t by w/v and u by z:

$$v^{-z-1} \int_0^v w^z (1-w/v)^v dw = \frac{z!v!}{(z+v+1)!}. \tag{109}$$

The integral exists if Re $z > -1$. Let us set $v = n =$ positive integer and take $w > 0$ to obtain

$$n^{-z-1} \int_0^n w^z (1-w/n)^n dw = \frac{z!n!}{(z+n+1)!}. \tag{110}$$

Now as $n \to \infty$ we have $(1-w/n)^n \to e^{-w}$ and this limit may be taken under the integral since $(1-w/n)^n < e^{-w}$ there. Thus we have

$$\lim_{n\to\infty} n^{z+1} \frac{z!n!}{(z+n+1)!} = \int_0^\infty w^z e^{-w} dw = z!, \tag{111}$$

or, since $(n+z+1)/n \to 1$,

$$\frac{(n+z)!}{n!n^z} \to 1, \quad n \to \infty. \tag{112}$$

We from (91) we have

$$\Gamma(z+1) = z! = \frac{\Gamma(z+n+1)}{(z+1)(z+2)\cdots(z+n)} = \frac{(z+n)!}{n!n^z} \frac{n!n^z}{(z+1)(z+2)\cdots(z+n)}, \tag{113}$$

and so, taking the limit $n \to \infty$ and using (112)

$$z! = \lim_{n\to\infty} \frac{n! n^z}{(z+1)(z+2)\cdots(z+n)}, \ \mathrm{Re}\, z > -1. \tag{114}$$

This representation of $z!$ was used as the definition of the factorial function by Gauss; it can be extended beyond $\mathrm{Re}\, z \ \geq -1$ as follows. We can rewrite (114) as

$$z! = \lim_{n\to\infty} \frac{n^z}{(1+z)(1+z/2)\cdots(1+z/n)}. \tag{115}$$

Taking the logarithm of both sides of (115), we have

$$\log z! = \lim_{n\to\infty} \Big[z\Big(\ln\frac{2}{1} + \ln\frac{3}{2} + \ldots + \ln\frac{n}{n-1} \Big) - \sum_{m=1}^{n} \log(1+z/m) \Big], \tag{116}$$

or

$$\log z! = \sum_{m=1}^{\infty} \Big[z \log\frac{m+1}{m} - \log\frac{m+z}{m} \Big] \tag{117}$$

This infinite series converges absolutely (see problem 5 of homework 10), and for z not a negative integer we obtain an absolutely convergent series of analytic functions, which defines an analytic function for z not a negative integer, and this the desired result.

Note that (114) implies that $\Gamma(z)$ *has no zeros.*

A variant of (114) can be optained using

$$lim_{n\to\infty}[\ln n - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}] = -\gamma, \tag{118}$$

where γ is Eulers constant $\approx .577215665$. Dividing out the factorial from the numerator of (114) and introducing exponential function suggested by (118) we obtain

$$z! = e^{-\gamma z} \prod n = 1^{\infty} \Big(1 + \frac{z}{n}\Big)^{-1} e^{z/n}. \tag{119}$$

4.3 Stirling's formula

We now derive the leading term of the asymptotic expansion of $\Gamma(z)$ for large $|z|$, using the method of steepest descent. We start with

$$\Gamma(z) = \int_C t^{z-1} e^{-t} dt = \int_C t^{-1} e^{\phi(t)} dt, \ \phi(t) = z \log t - t. \tag{120}$$

Here C will be a deformed path from the original path along the positive $\mathrm{Re}\, t$-axis. Now $\phi'(z) = 0$, $\phi''(z) = -1/z$, so the stationary point tends to infinity with z. So let's set $t = z(1+s)$, so the stationaryt point occurs at $s = 0$:

$$\Gamma(z) = \int_C (1+s)^{-1} z^z e^{-z} e^{z(\log(1+s)-s} ds. \tag{121}$$

Expanding,

$$\log(1+s) - s = -\frac{1}{2}s^2 + \frac{1}{3}s^3 - \ldots, \tag{122}$$

showing that the steepest path will be in the direction $\phi = -\frac{1}{2}\arg(z)$. Setting $s = e^{i\phi}u$, we then take u to be real. Finally, we set $u = v\sqrt{2/|z|}$, so that $s = v\sqrt{2/z}$, and let

$$\psi(w) = \log(1+w) - w + w^2/2. \tag{123}$$

Then we have

$$\Gamma(z) = \sqrt{2}z^{z-\frac{1}{2}}e^{-z}\int_C (1+v\sqrt{2/z})^{-1}e^{-v^2}\sqrt{2}e^{z\psi(v\sqrt{2/z})}dv. \tag{124}$$

We thus get to leading order

$$\Gamma(z) \sim \sqrt{2}z^{z-\frac{1}{2}}e^{-z}\int_{-\infty}^{+\infty} e^{-v^2}dv = \sqrt{2\pi}z^{z-\frac{1}{2}}e^{-z}[1+E], \tag{125}$$

as $z \to \infty$, where we note that $E = O(1/z)$. Stirling's formula (125) can be shown to hold for $|\arg\ z| < \pi$. The error E can be computed as a series by the above integral, but it is a bit awkward and other methods are better suited to. With a little work one can show from (124) that $E = \frac{1}{12z} + O(1/z^2)$.

Reference for this section: , Nehari's Introduction to Complex Analysis, pp. 245-248. See also Ahlfors pp. 198-206.

5 Lecture 13: The Riemann zeta function

This is defined by the series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{126}$$

This function is an important one in number theory. To give an example of its appearance there (we postpone the proof to later section) we note that

$$\zeta(z) = \prod_{p \in P}(1-p^{-z})^{-1}, \tag{127}$$

where P is the set of prime numbers.

The series converges absolutely if Re $z > 1$ and thus represents an analytic function in any closed subdomain of this region. Riemann showed that this function could be extended by analytic continuation outside this domain. To do this we transform the series into a definite integral on the positive real axis. For any positive integer n we see that

$$\int_0^\infty t^{z-1}e^{-nt}dt = \frac{1}{n^z}\int_0^\infty s^{z-1}e^{-s}ds = \frac{\Gamma(z)}{n^z}. \tag{128}$$

Thus

$$\Gamma(z)\sum_{n=1}^{m}\frac{1}{n^z}=\int_0^\infty t^{z-1}\sum_{n=1}^{m}e^{-nt}dt=\int_0^\infty\Big(\frac{e^{-t}-e^{-(m+1)t}}{1-e^{-t}}\Big)dt. \tag{129}$$

We now show that

$$I=\int_0^\infty \frac{t^{z-1}e^{-(m+1)t}}{1-e^{-t}}dt\to 0 \tag{130}$$

as $m\to\infty$. Indeed $1/(e^t-1)\le 1/t$ and since $|t^z|=t^x$ we have

$$|I|\le\int_0^\infty t^{x-2}e^{-mt}dt=\frac{\Gamma(x-1)}{m^{x-1}}. \tag{131}$$

Since we are assuming Re $z=x>1$, , the RHS tends to zero as $m\to\infty$.

Thus

$$\zeta(z)=\frac{1}{\Gamma(z)}\int_0^\infty\frac{t^{z-1}}{e^t-1}dt. \tag{132}$$

The point $t=0$ is the only singular point of the integrand and so we may replace the integral by an integral over the path of figure 3(b), provided we introduce the factor $(1-e^{2\pi iz})^{-1}$ and restrict R to be $<\ 2\pi$ (because of the denominator e^t-1):

$$\zeta(z)=\frac{1}{(1-e^{2\pi iz})\Gamma(z)}\int_C\frac{t^{z-1}}{e^t-1}dt. \tag{133}$$

Now observe that

$$(1-e^{2\pi iz})\Gamma(z)=-2ie^{\pi iz}\sin\pi z\ \Gamma(z)=\frac{-2\pi ie^{\pi iz}}{\Gamma(1-z)}, \tag{134}$$

where in the last step we have used the reflection identity in the form (99). This (133) is equivalent to

$$\zeta(z)=\frac{-e^{-\pi iz}\Gamma(1-z)}{2\pi i}\int_C\frac{t^{z-1}}{e^t-1}dt. \tag{135}$$

Now (135) provides the desired analytic continuation of $\zeta(z)$ to the complex plane except possibly for the singularities of $\Gamma(1-z)$, i.e. the positive integers. But we know that $\zeta(z)$ is analytic at $z=2,3,\ldots$, and at $z=\infty$, so the analytic extension is to all points of the extended complex plane excluding $z=1$, which is Riemann's result.

We show now that $\zeta(z)$ is indeed singular at $z=1$ and that it has a simple pole there with residue 1. Now when $z=1$ in the integral in (135) we have easily (cf. figure 3(b))

$$\int_c\frac{1}{e^t-1}dt=-2\pi i \tag{136}$$

since the deformed contours on the axis cancel out and only the contribution is the pole at $t=0$. Since $\Gamma(1)=1$ we see from (133) that the pole comes from the zero of $1 - e^{2\pi i z}$, so the derivative of this expression gives the residue as $-2\pi i/(-2\pi i) = 1$.

To show the significance of (135) we show now that

$$\sum_{n=1}^{\infty} n^2 = 0 \qquad (!!!). \tag{137}$$

Of course we have adopted the misleading notation of using the series where we mean the analytic extension (135). We see that

$$\zeta(-2) = -\frac{\Gamma(3)}{2\pi i} 2\pi i \mathrm{Res}_{t=0}(t^{-3}(e^t - 1)^{-1}) = 0, \tag{138}$$

as can be seen from the vanishing of the t^3 term in the Taylor series of $t(e^t - 1)^{-1}$ at $t = 0$. Of course (137) is no more of a shock than writing $\sum_{n=0}^{\infty} 2^n = -1$ when you mean $\frac{1}{1-z}\big|_{z=2}$.

5.1 Relation of $\zeta(z)$ to the prime numbers.

We now derive (127). First note the following result: An infinite product

$$\prod_{1}^{\infty} (1 + a_n) \tag{139}$$

converges, i.e.

$$\lim_{n\to\infty} \prod_{1}^{n} (1 + a_m) \tag{140}$$

exists, if and only if the series

$$\sum_{n=1}^{\infty} \mathrm{Log}(1 + a_n) \tag{141}$$

converges, where the principle branch of each term is taken. Convergence is always assume to be to a nonzero complex number.

It is clear by taking logarithms that convergence of the series insures convergence of the product. To prove that convergence of the product insures convergence of the series on has to think a little about convergence of the arguments of the logarithms. Let the product converge to $\mathcal{P} \neq 0$, the series to $\mathcal{S}$, and let the corresponding partial sums be $\mathcal{P}_n, \mathcal{S}_n$. Then

$$\mathrm{Log}\, \mathcal{P}_n = \mathcal{S}_n + k_n 2\pi i, \tag{142}$$

where $\{k_n\}$ is a sequence of integers. But Log $(\mathcal{P}_n/\mathcal{P}) \to 0$ and so

$$\mathcal{S}_n - \mathrm{Log}\, \mathcal{P} + k_n 2\pi i \to 0 \tag{143}$$

But then

$$\mathcal{S}_{n+1} - \mathcal{S}_n + (k_{n+1} - k_n)2\pi i = \text{Log}\,(1 + a_{n+1}) + (k_{n+1} - k_n)2\pi i \to 0. \quad (144)$$

But since $|\text{Arg}(1 + a_{n+1})| < \pi$, necessarily $k_{n+1} - k_n \to 0$ and so k_n must ultimately be a fixed integer k, and $\mathcal{S}_n$ must converge to Log $\mathcal{P} - 2\pi ik$.

An infinite product is said to converge absolutely if the corresponding series (141) converges absolutely. But since $a_n \to 0$ in any case, for n sufficiently large we have

$$(1 - \epsilon)|a_n| < |\log(1 + a_n) < (1 + \epsilon)|a_n|, \quad (145)$$

for some sufficiently small $\epsilon > 0$. Accordingly, *a necessary and sufficient condition for the absolute convergence of* (139) *is the convergence of* $\sum_{n=1}^{\infty} |a_n|$.

We may now assert that the infinite product

$$\frac{1}{\zeta(z)} = \prod_{p \in P} (1 - p^{-z}) \quad (146)$$

converges absolutely and a uniformly for $\Re(z) \geq x_0 > 1$ if the same is true of the series $\sum_{n=1}^{\infty} |p^{-z}| = \sum_{n=1}^{\infty} p^{-x}$. This series, involving the primes, is a sum over a subset of the positive integers, so its absolute convergence follows.

Now if $x > 1$ we see that

$$\zeta(z)(1 - 2^{-z}) = \sum_{n=1}^{\infty} (1 - 2^{-z})n^{-z} = \sum_{n=1}^{\infty} n^{-z} - \sum_{n=1}^{\infty} (2n)^{-z} = \sum_{m=0}^{\infty} (2m + 1)^{-z}. \quad (147)$$

That is, the sum is now over the set M_1 of odd positive integers. Similarly

$$\zeta(z)(1 - 2^{-z})(1 - 3^{-z}) = \sum_{m \in M_2} m^{-z} \quad (148)$$

where M_2 is the set of integers divisible by neither 2 or 3. Thus

$$\zeta(z)(1 - 2^{-z})(1 - 3^{-z}) \cdots (1 - p_N^{-z}) = \sum_{m \in M_N} m^{-z} \quad (149)$$

where now the sum is over m not divisible by $2, 3, 5, 7, \ldots, p_N$, i.e. the first N prime numbers greater than 1. If there were only a finite number of primes, for some N the RHS of (149) would be 1, since that is the first and only term in that case, and so we would have that $\zeta(1) =$ finite number. This contradicts the fact that $z = 1$ is a simple pole of $\zeta(z)$. Thus there are an infinite number of primes p_n and $p_n \to \infty$ with n.

Looking again at the RHS of (149) we see that the second term is $p^{-(N+1)z}$, followed by other terms of $n^{-z}, n > p_{N+1}$. Summing the "tail " of the series we have by Cauchy's test that the absolute convergence implies, if $\Re(z) > 1$, that

$$\lim_{N \to \infty} \zeta(z)(1 - 2^{-z})(1 - 3^{-z}) \cdots (1 - p_N^{-z}) = 1, \quad (150)$$

which is equivalent to (127).

5.2 A functional equation for $\zeta(z)$.

Riemann found a beautiful equation relating $\zeta(z)$ and $\zeta(1-z)$:

$$\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2)\Gamma(1-z)\zeta(1-z). \tag{151}$$

Like the reflection identity of the gamma function, this relation allows values of ζ in the halfplane $x < 0$ to be accessed easily.

To establish (151) we consider in (135) first, the path C as in figure 3(b) with $R < 2\pi$, as we used in (135), but also the same expression taken with the path C_n involving the radius $R = R_n$, where $2\pi n < R < 2\pi(n+1)$, see figure 5.

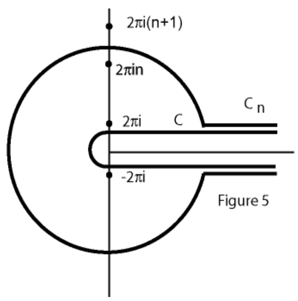


Figure 5

Call the corresponding RHSs of (135) I, I_n. Thus $I = \zeta(z)$ and

$$I - I_n = -e^{-\pi i z}\Gamma(1-z) \times \sum_{Res} \frac{t^{z-1}}{e^t - 1}, \tag{152}$$

where the sum is over the residues at $t = 2\pi i k, k = \pm 1, \ldots, \pm n$.

We first show that $I_n \to 0$ as $n \to \infty$. We need to show that

$$\lim_{n\to\infty} \int_{C_n} \frac{t^{z-1}}{e^t - 1} dt = 0. \tag{153}$$

Indeed over the circular part of C_n we not that, e.g. taking the radius as $R_n = 2\pi n + \pi$, we see that t stays a finite distance from $2\pi n i$ and $2\pi(n+1)i$ and so there is a lower bound on $|e^t - 1|$ on C_n which is independent of n. Thus

$$\Big| \int_{|t|=R_n} \frac{t^{z-1}}{e^t - 1} dt \Big| \le K R_n^x \tag{154}$$

for some constant K independent of n, and so this contribution tends to zero with n if $x < 0$. The integrals along the real(t) axis clearly also tend to zero with n. We thus have

$$\zeta(z) = -e^{-\pi i z}\Gamma(1-z) \sum_{n=1}^{\infty} [(2\pi i n)^{z-1} + (-2\pi i n)^{z-1}]. \tag{155}$$

Now

$$-e^{-\pi i z}[(2\pi i n)^{z-1} + (-2\pi i n)^{z-1}] = e^{-\pi i (z-1)}[(2\pi i n)^{z-1} + (-2\pi i n)^{z-1}]$$

$$= [(-2\pi i n)^{z-1} + (2\pi i n)^{z-1}] = (2\pi n)^{z-1} 2\sin(\pi z/2). \quad (156)$$

Thus using this in (155) we have

$$\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2)\Gamma(1-z)\zeta(1-z) \quad (157)$$

as claimed, provided that $x < 0$. But since both sides agree in this domain, the RHS is the same analytic function as the LHS of (157) , the latter being define for all $z \neq 1$. At $z = 1$, both are infinite, so in fact (157) is valid for all finite z.

5.3 The zeros of $\zeta(z)$.

We first note that the points $z = -2n, n = 1, 2, \ldots$ are zeros of $\zeta(z)$. Indeed we observe that

$$g(z) = \frac{1}{e^z - 1} + \frac{1}{2} \quad (158)$$

is an odd function of z, since

$$g(-z) = -\frac{e^z}{e^z - 1} + \frac{1}{2} = -\frac{1}{e_z - 1} - \frac{1}{2}. \quad (159)$$

Thus in (135), with $z = -2n$, we see that $t^{-2n-1}\left[\frac{1}{e^t-1} + \frac{1}{2}\right]$ is even and so cannot have a non-zero residue if $n = 1, 2, \ldots$. The zeros are called the *trivial* zeros of $\zeta(z)$.

From (127) it follows that there can be no zeros of $\zeta(z)$ in $x > 1$. Then (recalling that the gamma function has no zeros), (157) implies that ζ has no non-trivial zeros in $x < 0$. Thus all non-triviaql zeros of ζ must lie in the strip $0 \leq x \leq 1$. The famous *Riemann conjecture* states that all zeros of ζ lie on the line $x = 1/2$. It is probably the most famous unproved conjecture in mathematics.

References for this section: Nehari pp. 248-251, Ahlfors pp. 212-218.

6 Lecture 14: Transform methods

6.1 The Fourier transform

For a given real function $f(x)$ we define the Fourier transform of f, written $\mathcal{F}f$, by

$$\mathcal{F}f = \int_{-\infty}^{+\infty} e^{-ikx} f(x) dk \equiv \hat{f}(k). \quad (160)$$

Here x, k are real variables. We have studied how residue theory can be used to evaluate integrals of this type. For example

$$\mathcal{F}\frac{1}{\pi}\frac{1}{1+x^2} = -2i\text{sgn}(k)\ \text{Res}_{z=-\text{sgn}(k)i}\frac{e^{-ikz}}{1+z^2} = e^{-|k|}. \quad (161)$$

Fourier transforms are the natural continuous generalization of Fourier series, with the function $\hat{f}(k)$ analogous to the Fourier coefficients of the series.

We now want to study this as a map from functions of x to functions of k. In particular we would like to define and study the inverse map $\mathcal{F}^{-1}$ with the property that $\mathcal{F}^{-1}\hat{f}(k) = f(x)$. It turns out that

$$\mathcal{F}^{-1}\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\hat{f}(k)dk. \tag{162}$$

Before we show this let us verify it for our example:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}e^{-|k|}dk = \frac{1}{\pi}\int_{0}^{\infty} \cos kx \; e^{-k}dk = \frac{1}{\pi}\frac{1}{1+x^2} \tag{163}$$

by a well known definite integral of the calculus. Introducing a parameter $\epsilon > 0$, we have easily the related function $f_\epsilon(x) = \frac{1}{\epsilon\pi}\frac{1}{1+(x/\epsilon)^2}$. Then we have

$$\mathcal{F}f_\epsilon = e^{-\epsilon|k|} \equiv \hat{f}_\epsilon(k), \; \mathcal{F}^{-1}\hat{f}_\epsilon(k) = f_\epsilon(x). \tag{164}$$

We now want to verify (162) for arbitrary functions. Of course we must deal with function for which the Fourier integrals exist, and usually one deals with the Hilbert space of complex-valued functions f which are square integrable, i.e. such that

$$\int_{-\infty}^{+\infty} |f(x)^2 dx \quad < \infty. \tag{165}$$

We want to make use of the transform pair $f_\epsilon(x), \hat{f}_\epsilon(k)$ to formally verify (162), but do this in the limit of $\epsilon \to 0$. In this case $f_\epsilon(x)$ tends to a distribution,

$$\delta(x) = \lim_{\epsilon\to 0} f_\epsilon(x). \tag{166}$$

It has the property that

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0. \end{cases} \tag{167}$$

Also, for any $c > 0$,

$$\int_{-c}^{c} \delta(x)dx = 1. \tag{168}$$

These and the results to follow by interpreting $\delta(x)$ as a limit. We also note that $\delta(-x) = \delta(x)$ and that, if $f(x)$ is a smooth function and $a < x_0 < b$, then

$$\int_a^b f(x)\delta(x-x_0)dx = \int_a^b f(x)\delta(x_0-x)dx = f(x_0). \tag{169}$$

The $\epsilon \to 0$ limit thus gives us the following formal identities:

$$\mathcal{F}\delta(x) = \int_{-\infty}^{+\infty} \delta(x)e^{-ikx}dx = 1, \; \mathcal{F}^{-1}1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dx = \delta(x). \tag{170}$$

We now consider the Fourier transform and its inverse, freely interchanging the order of integration and using (170):

$$\begin{aligned}\mathcal{F}^{-1}\hat{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\int_{-\infty}^{+\infty} e^{-ikx'}f(x')dx'dk \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{2\pi}e^{ik(x-x')}dkf(x')dx' \\ &= \int_{-\infty}^{+\infty}\delta(x-x')f(x')dx' = f(x). \end{aligned} \tag{171}$$

Again, one can justify these arguments by realizing that

$$\delta(x) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty} e^{ikx-\epsilon|k|}dk \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dk. \tag{172}$$

6.2 The Laplace transform

In many applications we are in interested in the future behavior of a system starting from some initial state at time zero. Any quantitative function of time describing this system, $f(t)$ say, can be viewed as $= 0$ for $t < 0$ and having the initial value $f(0)$. The object is then to determine the future behavior of the system $f(t), t > 0$. In such cases the Fourier transform and its inverse may be reconfigured in a form due to Laplace. In the Fourier analysis we shall replace x by t and $f(x)$ by $e^{-ct}f(t)$, where c is a positive real number. The purpose of this factor will be apparent presently. It is assumed that $f(t) = 0, t < 0$. Then the statement $f(x) = \mathcal{F}^{-1}\mathcal{F}f(x)$ is converted to

$$e^{-ct}f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikt}\int_{0}^{\infty} e^{-ikt'}e^{-ct'}f(t')dt'dk. \tag{173}$$

This can be written

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(c+ik)t}\int_{0}^{\infty} e^{-(c+ik)t'}f(t')dt'dk. \tag{174}$$

Let us define $s = c + ikt$. We then rewrite (174) as

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\int_{0}^{\infty} e^{-st'}f(t')dt'ds. \tag{175}$$

It is natural now to consider function $f(t)$ which are *of exponential order* in the sense that

$$\int_{0}^{\infty} e^{-ct}|f(t)|dt < \infty. \tag{176}$$

For such functions we define the Laplace transform of $f(t)$ as

$$\hat{f}(s) = \int_{0}^{\infty} e^{-st}f(t)dt \equiv \mathcal{L}f. \tag{177}$$

Given (176) if follows from (177) that $\hat{f}(s)$ is an analytic function of s for $\mathrm{Re}(s) \geq c$. Hence all singularities of $\hat{f}(s)$ lie in the region $\mathrm{Re}(s) < c$. We see then from (175) that the inverse Laplace transform $\mathcal{L}^{-1}$ is defined by

$$\mathcal{L}^{-1}\hat{f}(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\hat{f}(s)ds = f(t). \tag{178}$$

The path taken by s in this integral, called the *Bromwich path*, is on the line $\mathrm{Re}(s) = c$, so is on a line to the right of all singularities of $\hat{f}(s)$. This means the natural closure of the path, on a large circular arc to the left of this line for example, will encompass the singularities and the contour integral will usually be evaluated by calculating residues.

6.3 Convolutions

In the evaluation of Fourier and Laplace transforms in applications one frequently arrives as a function of k or s which is a product of two functions. In that case it is convenient to have a formula for the inverse transform that involves the (presumably simpler) inverses of the two constituent functions.

Consider first the Fourier transform. We define the *convolution product* of two functions $f(x), g(x)$ by

$$(f * g)(x) = \int_{-\infty}^{+\infty} g(x')f(x-x')dx'. \tag{179}$$

Then

$$\mathcal{F}(f * g)(x) = \int_{-\infty}^{+\infty} e^{-ikx}\Big[\int_{-\infty}^{+\infty} g(x'))f(x-x')dx'\Big]dx$$

$$= \int_{-\infty}^{+\infty} e^{ikx'}g(x'dx'\Big[\int_{-\infty}^{+\infty} e^{-k(x-x')}f(x-x')dx\Big] = \hat{g}(k)\hat{f}(k). \tag{180}$$

Thus we have our desired formula for inverting a product:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\hat{f}(k)\hat{g}(k)dk = \mathcal{F}^{-1}[\hat{f}(k)\hat{g}(k)] = \int_{-\infty}^{+\infty} g(x')f(x-x')dx'. \tag{181}$$

Note that with a change of variable the RHS of (181) can also be written

$$\int_{-\infty}^{+\infty} g(x-x')f(x')dx'. \tag{182}$$

A special case of (181) (in the form (182)), called the *Parseval formula*, is obtained by putting $g(x) = \bar{f}(-x)$, $\bar{f}$ being the complex conjugate, and setting $x = 0$:

$$\int_{-\infty}^{+\infty} f(x)\bar{f}(x)dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(k)\hat{g}(k)dk. \tag{183}$$

Here

$$\hat{g}(k) = \int_{-\infty}^{+\infty} e^{-ikx}\bar{f}(-x)dx = \int_{-\infty}^{+\infty} e^{ikx}\bar{f}(x)dx = \overline{\hat{f}(k)}. \qquad (184)$$

Thus the Parseval formula reduces to

$$\int_{-\infty}^{+\infty} |f|^2(x)dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{f}|^2(k)dk. \qquad (185)$$

The analogous convolution result for the Laplace transform uses

$$(f * g)(t) = \int_0^t g(t')f(t-t')dt'. \qquad (186)$$

We then have

$$\mathcal{L}(f * g) = \int_0^\infty e^{-st}\Big[\int_0^t g(t')f(t-t')dt'\Big]dt \qquad (187)$$

Here the sector $t' \geq t < \infty$ is being covered by first integrating t' from 0 to t. But we may convert this by first integrating with respect to t from t' to ∞.We then obtain

$$\mathcal{L}(f * g) = \int_0^\infty g(t')\Big[\int_{t'}^\infty e^{-st}f(t-t')dt\Big]dt'$$

$$= \int_0^\infty g(t')e^{-st'}\Big[\int_{t'}^\infty e^{-s(t-t')}f(t-t')dt\Big]dt'$$

$$= \int_0^\infty g(t')e^{-st'}dt'\Big[\int_0^\infty e^{-su}f(u)du\Big] = \hat{f}(s)\hat{g}(s), \qquad (188)$$

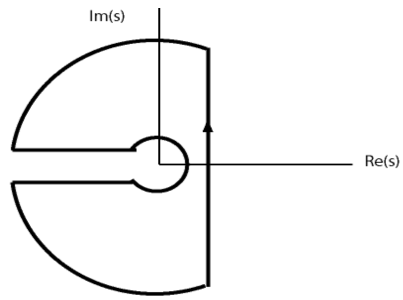
or

$$\mathcal{L}^{-1}[\hat{f}(s)\hat{g}(s)] = \int_0^t g(t')f(t-t')dt'. \qquad (189)$$

As an example, consider the inverse Laplace transform of s^{-a}, where $0 < a < 1$. We take $c > 0$ and close the contour by a large circular arc to the left, integrals along both sides of the branch cut, and a small circle around the origin, see figure 6. By the usual methods we can show that when $t > 0$ the circles may be taken to their limits with zero contribution, so the integral along the Bromwich path is

$$\mathcal{L}^{-1}s^{-a} = \frac{1}{2\pi i}\int_0^\infty e^{-ut}u^{-a}du[e^{i\pi a} - e^{-i\pi a}] = \frac{\sin \pi a}{\pi}t^{a-1}\Gamma(1-a). \qquad (190)$$

Figure 6:



6.4 Applications

6.4.1 The IVP for an inhomogeneous linear ODE with constant coefficients.

We first look at the Laplace transform of df/dt: By integration by parts, assuming that $\lim_{t\to\infty} e^{-st} f(t) = 0$

$$\mathcal{L}f' = -f(0) + s\hat{f}(s). \tag{191}$$

Similarly assuming that f' and higher derivatives satisfy the same limit condition as f,

$$\mathcal{L}f'' = -f'(0) + s\mathcal{L}f'(t) = -f'(0) - sf(0) + s^2\hat{f}(s), \tag{192}$$

$$\mathcal{L}f''' = -f''(0) - sf'(0) - s^2 f(0) + s^3\hat{f}(s), \tag{193}$$

and so on.

Now consider the ODE

$$a_n \frac{d^n f}{dt^n} + a_{n-1}\frac{d^{(n-1)}f}{dt^{(n-1)}} + \ldots + a_1\frac{df}{dt} + a_0 f = A(t). \tag{194}$$

Here $A(t)$ is given. Let

$$f(0) = b_0, df/dt(0) = b_1, \ldots, df^{n-1}/dt^{n-1}(0) = b_{n-1}. \tag{195}$$

Under the above limit conditions the Laplace transform of (194), is of the form

$$(a_n s^n + a_{n-1}s^{n-1} + \ldots + sa_1 + a_0)\hat{f}(s) \equiv P_n(s)\hat{f}(s) = Q_{n-1}(s) + \hat{A}(s), \tag{196}$$

where Q_{n-1} is a polynomial of degree $n-1$ determined by the initial values $b_0, \ldots, b_{n-1}$. Thus

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{Q_{n-1}(s) + \hat{A}(s)}{P_n(s)}ds. \tag{197}$$

Note that, by choosing c so that (i) all zeros of P_n lie to the left of $\mathrm{Re}(s) = c$, and (ii) $\hat{A}(s)$ exists, we insure that $f(t)$ satisfies (176). Evaluation of the integral by residue theory then constructs the solution to the problem.

Example 1. Consider

$$f_{tt} + f_t = 1, f(0) = 1,\ f(0) = 1, f'(0) = 2. \tag{198}$$

The Laplace transform leads to

$$-2 - s + s^2\hat{f} - 1 + s\hat{f} = \frac{1}{s}. \tag{199}$$

Thus

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{1+3s+s^2}{s^2(1+s)}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Big[\frac{1+2s}{s^2} - \frac{1}{1+s}\Big]ds. \tag{200}$$

Choosing $c > 0$ and calculating residues we obtain

$$f(t) = t + 2 - e^{-t}. \tag{201}$$

Example 2. Consider

$$\frac{d^3f}{dt^3} + 4\frac{df}{dt} = e^t,\ f(0) = 0, \frac{df}{dt}(0) = 1. \frac{d^2f}{dt^2}(0) = -1. \tag{202}$$

The Laplace transform gives, assuming $c > 1$,

$$1 - s + (s^3 + 4s)\hat{f}(s) = -\frac{1}{1-s}. \tag{203}$$

Consequently

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{-s^2+2s-2}{s(1-s)(4+s^2)}ds. \tag{204}$$

Calculating residues we obtain

$$f(t) = \frac{1}{5}e^t - \frac{1}{2} + \frac{3}{10}\cos 2t + \frac{2}{5}\sin 2t. \tag{205}$$

6.4.2 The IVP for the one-dimensional heat equation.

The one dimensional heat equation is the following PDE in x and t.

$$u_t - u_{xx} = 0. \tag{206}$$

We seek to solve the initial value problem for $-\infty < x < +\infty$ with $u(x, 0) = u_0(x)$. It is natural, given the infinite interval, to use the Fourier transform:

$$\mathcal{F}(u_t - u_{xx}) = \hat{u}_t + k^2\hat{u} = 0. \tag{207}$$

Solving and using the initial condition in Fourier space,

$$\hat{u} = \hat{u}_0(k)e^{-k^2t}. \tag{208}$$

Now

$$\mathcal{F}^{-1}e^{-k^2t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-k^2t}dk. \tag{209}$$

We change to the variable $s = k\sqrt{t} - ix/(2\sqrt{t})$, which amounts to a shift to a new contour parallel to the Re(k) axis:

$$\mathcal{F}^{-1}e^{-k^2t} = \frac{e^{-x^2/(4t)}}{2\pi\sqrt{t}}\int_{-\infty}^{+\infty} e^{-s^2}ds = \frac{1}{\sqrt{4\pi t}}e^{-x^2/(4t)} \equiv U(x,t). \tag{210}$$

The function $U(x,t)$ is the fundamental solution of the heat equation, representing physically the temperature of a bar which initially contained a unit amount of heat concentrated at $x = 0$.

It follows from the convolution formula that

$$\mathcal{F}^{-1}\hat{u}_0(k)e^{-k^2t} = \int_{-\infty}^{+\infty} u_0(y)U(x-y,t)dy, \tag{211}$$

which is the Poisson representation of the solution to the initial-value problem for the heat equation on $-\infty < x < +\infty$.

Any initial-value problem suggests use of the Laplace transform, and it is instructive to see how this works in the present example. We have, given the initial condition $u(x,0) = u_0(x)$,

$$\mathcal{L}(u_t - u_{xx}) = -u_0(x) + s\hat{u} - \hat{u}_{xx} = 0. \tag{212}$$

The Green's function for the operator $s - \frac{\partial^2}{\partial x^2}$ is $G(x,y) = \frac{1}{2\sqrt{s}}e^{-\sqrt{s}|x-y|}$ so the solution of (212) is

$$\hat{u}(x,s) = \frac{1}{2\sqrt{s}}\int_{-\infty}^{+\infty} u_0(y)e^{-\sqrt{s}|x-y|}dy. \tag{213}$$

Now consider

$$\mathcal{L}^{-1}\Big[\frac{1}{2\sqrt{s}}e^{-\sqrt{s}|x|}\Big] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Big[\frac{1}{2\sqrt{s}}e^{-\sqrt{s}|x|}\Big]ds \tag{214}$$

We take the negative Re(s) axis for the branch cut of $\sqrt{s}$ and use the contour of figure 6. We then get

$$\frac{1}{2\pi i}\Big[-\frac{1}{2i}\int_0^{\infty} s^{-1/2}e^{-i|x|\sqrt{s}-st}ds - \frac{1}{2i}\int_0^{\infty} s^{-1/2}e^{-i|x|\sqrt{s}-st}ds\Big]$$

which, with $s = k^2$, returns us to the result of use of the Fourier transform:

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i|x|k-tk^2}du. \tag{215}$$

From (213) we thus have as before

$$\mathcal{L}^{-1}\hat{u}(x,s) = \int_{-\infty}^{+\infty} u_0(y)U(x-y,t)dy. \tag{216}$$

6.4.3 The IVP for the one-dimensional wave equation.

If the wave speed is normalized to unity, the one-dimensional wave equation is

$$u_{tt} - u_{xx} = 0. \tag{217}$$

We consider the initial-value problem

$$u(x,0) = f(x), u_t(x,0) = g(x). \tag{218}$$

The Fourier transform gives

$$\mathcal{F}[u_{tt} - u_{xx}] = \hat{u}_{tt} + k^2\hat{u} = 0. \tag{219}$$

In Fourier space the solution of the initial-value problem for the second-order ODE (219) is

$$\hat{u}(k,t) = \hat{f}(k)\cos kt + \hat{g}(k)\frac{\sin kt}{k}. \tag{220}$$

But

$$\mathcal{F}^{-1}\cos kt = \frac{1}{4\pi}\int_{-\infty}^{+\infty}[+e^{ik(x+t)} + e^{ik(x-t)}]dk = \frac{1}{2}[\delta(c+t) + \delta(x-t)], \tag{221}$$

$$\begin{aligned}\mathcal{F}^{-1}k^{-1}\sin kt &= \mathcal{F}^{-1}\frac{1}{2ki}\left[e^{ik(x+t)} - e^{ik(x-t)}\right] \\ &= \frac{1}{2}[\text{sgn }(x+t) - \text{sgn }(x-t)]. \end{aligned} \tag{222}$$

The convolution formula thus gives

$$\begin{aligned} u = \frac{1}{2}\int_{-\infty}^{+\infty} &\Big[f(y)[\delta(x-y+t) + \delta(x-y-t)] \\ &+g(y)[\text{sgn }(x-y+t) - \text{sgn }(x-y-t)]\Big]dy. \end{aligned} \tag{223}$$

The term involving f gives $\frac{1}{2}[f(x-t) + f(x+t)]$ by the basic property of $\delta(x)$. For the term involving g, note that sgn $(x-y+t)$ − sgn $(x-y-t)$ vanishes unless $x-t < y < x+t$, when it equals 2. Thus we obtain the D'Alembert solution of the IVP of the wave equation,

$$u(x,t) = \frac{1}{2}[f(x-t) + f(x+t)] + \int_{x-t}^{x+t} g(y)dy. \tag{224}$$

Finally, consider the following initial-boundary value problem for the wave equation (217) on $0 < x < L$:

$$u(x,0) = 0,\ u_t(x,0) = 0,\ u(0,t) = 0,\ u(L,t) = 1. \tag{225}$$

Thus problem arises e.g. when a stretched elastic string of length $2L$, initially at rest, suddenly undergoes a displacement of it's middle point. By symmetry we can consider only the domain $0 < x < L$ with the conditions (225).

The Fourier transform is ill suited for finite intervals, but the Laplace transform can be used. Since the initial conditions are null we have

$$s^2\hat{u} - u_{xx} = 0. \tag{226}$$

The solutions of this equation must satisfy

$$\hat{u}(0,s) = 0,\ \hat{u}(L,s) = \mathcal{L}\ 1 = \frac{1}{s}. \tag{227}$$

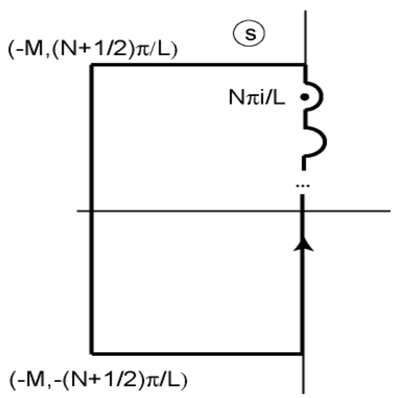
Thus

$$\hat{u}(x,s) = \frac{\sinh sx}{s\sinh sL}. \tag{228}$$

Thus

$$u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Big[\frac{\sinh sx}{s\sinh sL}\Big]ds. \tag{229}$$

For the residue calculation we take the contour show in Figure 7.



On the upper horizontal line, we have $s = \sigma + \pi(N+1/2)i;\ -M \le \sigma \le 0$ and so

$$\Big|e^{st}\frac{\sinh sx}{s\sinh sL}\Big| \le \frac{L}{\pi N}\frac{\cosh \sigma x}{\cosh \sigma L} \le \frac{L}{\pi N} \tag{230}$$

since $x \le L$. The length of the line is M, by taking$M = \sqrt{N}$ say, we get zero contribution in the limit $N \to \infty$. Similarly for the lower horizontal line it will have $\mathrm{Im}(s) = -(N+1/2)\pi i$ and there will be no contribution in the limit. On the left vertical line we have

$$\Big|e^{st}\frac{\sinh sx}{s\sinh sL}\Big| \le e^{-Mt}(2N+1)/M \to 0, \tag{231}$$

as $N \to \infty$ for any $t > 0$.Thus the inversion contour integral is equal to the sum of the residues at $\mathrm{Im}(s) = n\pi i/L, n = 0, \pm 1, \pm 2, \ldots$. We then have

$$u(x,t) = x/L + \sum_{n=1}^{\infty}\Big[\frac{1}{\pi n i}e^{in\pi t/L}\frac{\sinh(in\pi x/L)}{\cosh(in\pi)} + \frac{1}{\pi n i}e^{-in\pi t/L}\frac{\sinh(in\pi x/L)}{\cosh(in\pi)}\Big], \tag{232}$$

This yields

$$u(x,t) = x/L + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi t}{L}\right). \tag{233}$$

Reference for this section: Ablowitz and Fokas, pp. 267-300.