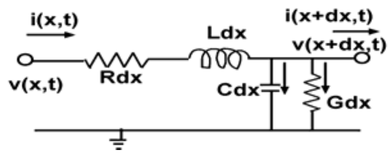
Applied Mathematics II **PROBLEM SET 1** **Due February 6, 2003**

1. The *telegrapher's equation* is intended to model a coaxial electrical transmission line. The current-carrying wire core is coupled by resistance and capacitance to the grounded covering. The continuous properties of the cable can be represented by a circuit diagram involving differential amounts of resistance R, capacitance C, inductance L, and impedance, $G = 1/R$.



Current i and voltage v relations for these elements are $v = iR$ for the resistance, $i = Cdv/dt$ for the capacitor, and $v = Ldi/dt$ for the inductance. Explain how the circuit leads to the following equations:

$$i(x,t) = i(x+dx,t) + Cdx\,\frac{\partial v}{\partial t} + Gdx\; v,$$

$$v(x,t) = v(x+dx,t) + Rdx\; i + Ldx\,\frac{\partial i}{\partial t}$$

Express these as PDEs and combine to obtain a single equation for i. What can be said about the special case $R = G = 0$? About the special case $L = G = 0$?

2. Problem 5.2, page 169 of text.

3. In one-dimensional gas dynamics, let the gas particle initialy at $x = a$ have the Lagrangian coordinate $x(a,t)$, and write $x_t = \frac{\partial x}{\partial t}$ holding a fixed. Then the Eulerian velocity field $u(x,t)$ satisfies

$$x_t = u(x(a,t),t).$$

From this and using the chain rule verify that Lagrangian acceleration x_{tt} of the particle has the Eulerian form given by

$$x_{tt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}.$$

Generalize this result to higher dimension, with $\mathbf{x}_t = \mathbf{u}(\mathbf{x},t)$ where $\mathbf{x}, \mathbf{u}$ are n-vectors.

4. Find the solution of $u_{tt} - c^2 u_{xx} = 0$ when

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases},\; g(x) = 0.$$

(Simply express it in terms of f.) Assuming $c = 1$, sketch $u(x,0), u(x,1), u(x,2)$.

5. Consider the solution of $u_{tt} - c^2 u_{xx} = 0$ in $x \ge 0, t \ge 0$ satisfying $u(x,0) = 0, u_t(x,0) = 0, u(0,t) = h(t)$, h a twice continuously differentiable function satisfying $h(0) = 0$.

a) $u(x,t)$ will then vanish for $0 \le t \le x/c$. Why?

b) What is a continuous solution of this problem, in terms of h? (Hint: Try finding a solution by inspection. Then verify by assuming $u = F(x+ct) + G(x-ct)$ and using (a).)

6. Consider the solution of the inhomogeneous wave equation $u_{tt} - c^2 u_{xx} = F(x,t)$ in the entire $x - t$ plane, with $F(x,t) = 0, t < 0$, satisfying null conditions $u(x,0) = u_t(x,0) = 0$.

a) Introduce *characteristic coordinates* $\alpha = x + ct, \beta = x - ct$. Show that in these coordinates the problem may be written in the $\alpha - \beta$ plane as

$$\frac{\partial^2 u}{\partial \alpha \partial \beta} \equiv u_{\alpha\beta} = \frac{-1}{4c^2} F(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2c}).$$

b) By integration, show that for integrable F the double integral

$$u(\alpha, \beta) = \frac{1}{4c^2} \int_\beta^\infty \int_{-\infty}^\alpha F(\frac{\alpha'+\beta'}{2}, \frac{\alpha'-\beta'}{2c}) d\alpha' d\beta'.$$

satisfies the null conditions as required.

c) Change variable back to x, t in the last solution and show that

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-t')}^{x+c(t-t')} F(x', t') dx' dt'.$$

Show that this can be written

$$u(x,t) = \frac{1}{2c} \int \int_\Delta F(x', t') dx' dt',$$

where Δ is the triangle of the $x - t$ plane formed by the characteristics through (x, t) and the x-axis.

d) Verify by differentiation that this solves the inhomogeneous equation we started with.

e) Using the result of the last problem, find the D'Alembert form of the solution of the inhomogeneous IVP: $u_{tt} - c^2 u_{xx} = F(x,t)$, $u(x,0) = f(x), u_t(x,0) = g(x)$, by adding the solutions of two problems.