Applied Mathematics II **PROBLEM SET 2** **Due February 13, 2003**

1. Consider the telegrapher's system

$$i_x + Cv_t + Gv = 0,\ v_x + Li_t + Ri = 0$$

on the domain $0 \le x \le l, t > 0$. Multiply the first equation by v, the second by i, and add to get

$$(iv)_x + C(v^2/2)_t + L(i^2/2)_t + Gv^2 + Ri^2 = 0.$$

Given initial conditions $i(x,0) = i_0(x), c(x,0) = v_0(x), 0 \le x \le l$ show that there is at most one solution to the IBVP with these initial conditions and certain boundary conditions on $x = 0, l$. What are the possible boundary conditions where this energy method works? (You may use the following differential inequality: If $y(x) \ge 0$ is differentiable with $dy/dx \le -\alpha y,\ \alpha > 0$, for all $x \ge 0$, then $y(x) \le y(0)e^{-\alpha x}$ for all $x \ge 0$.)

2. (a) Show that the sequence of integrable functions $f_n(x) = n^{-1}|x|^{\frac{1}{n}-1} \to 2\delta(x)$ as $n \to \infty$, as follows: Show that for every test function ϕ and $\epsilon = n^{-\alpha},\ \alpha > 0$,

$$\int_{-\infty}^{+\infty} \phi(x) f_n(x) dx = \int_{-\infty}^{-\epsilon} \phi(x) f_n(x) + \int_{-\epsilon}^{+\epsilon} \phi(x) f_n(x) dx + \int_{+\epsilon}^{\infty} \phi(x) f_n(x) dx$$

$$\to 0 + 2\phi(0) + 0,\ n \to \infty.$$

(Note that α must be appropriately chosen. Also recall that any test function has continuous derivatives of all orders for all x and vanishes outside a finite interval.)

(b) Show that $x\delta'(x) = -\delta(x)$ where δ' is the derivative of $\delta(x)$.

3. Problem 8.7 of text, page 316. Note that in this problem $c = 1$ and $L = \pi$.

5. Find solutions of the IBVP given by

$$u_{tt} - c^2 u_{xx} = 0,) < x < L, t > 0,$$

$$u(0,t) = 0, u(L,t) + u_x(L,t) = 0, t \ge 0, u_t(x,0) = 0$$

in the variables-separated form $u = T(t)X(x)$.

(a) Show X satisfies $X'' - KX = 0, X(0) = 0, X(L) + x'(L) = 0$ and argue that $K = -\lambda^2$ for real λ.

(b)Show that the eigenvalue λ satisfies $\tan(\lambda L) = -\lambda$. Graph $\tan z$ and $-z/L$ versus z and indicate roughly the eigenvalues on this graph. Using a calculator or Newton's method calculate the first four eigenvalues λ_n to three or four decimal places. Indicate why $\lambda_n L \approx (2n-1)\pi/2$ for large n and a fixed L;

5. Show that sound waves may be treated using a potential ϕ satisfying the wave equation $\phi_{tt} - c^2\nabla^2\phi = 0$ with velocity $\mathbf{u} = \nabla\phi$ and pressure $p = -\phi_t/\rho_0$. Here $c^2 = dp/d\rho(\rho_0)$. Find the potential of the three-dimensional, spherically symmetric sound field emitted by an oscillating sphere. Assume oscillations are of amplitude a about a radius r_0, where $a << r_0$, so the boundary condition to be satisfied is

$$\frac{\partial \phi}{\partial r}(r_0, 0) = a \sin \omega t$$

approximately. (Why is the last expression approximately correct?) Assume also that only outgoing waves appear in the solution. From your expression for ϕ compute the radial velocity field $u_r = \frac{\partial \phi}{\partial r} = \phi_r$. Your expression will involve the dimensionless parameter $\Omega = \omega r_0/c$. (The frequency of the standard note A above middle C is 440 Hertz, or cycles per second, in which case $\omega = 880\pi$. Taking $r_0 = 1cm$ and $c = 350m/sec$ we get $\omega r_0/c \approx .08$. We may take the parameter to be of order unity.) The length scale $R = c/\omega \approx$ 12cm separates the "near" and "far" fields of the sound pattern. Discuss the r-variation of ϕ_r in the two cases $r_0 + a < r << R$ and $r >> R$.

6. Establish uniqueness of the solution of the IVP for the wave equation in N dimensions using the energy method, in the following case:

(a) Show that if u is twice continuously differentiable and $u_{tt} - c^2\nabla^2 u = 0$, for $t > 0$, all $\mathbf{x}$, with $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_N^2}$, and also either u or u_t is identically zero outside a sphere of some large radius R centered at the origin, then the energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} (u_t^2 + c^2(\nabla u)^2) dx_1 \ldots dx_N$$

is constant in time. Use the fact that then the range of influence of the initial data lies inside the sphere or radius $R + ct$ at time t.

(b) Then obtain uniqueness by considering the null initial conditions $u_t = 0, u = 0$.