Fluid Dynamics I PROBLEM SET 4 Due October 13, 2003

1. (Reading: Batchelor 543-545). Consider axisymmetric motion, with velocity $\mathbf{u} = (u_r, u_\theta, u_z)$, of a fluid of constant density. The equation of continuity in cylindrical polar coordinates is

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial r u_r}{\partial r} = 0.$$

(a) Show that this equations is satisfied if

$$u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z},$$

for some function ψ, and verify that

$$\omega_\theta = -\frac{1}{r}L(\psi), L = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}.$$

(b) Assume the flow is *steady*. i.e. $\partial \mathbf{u}/\partial t = 0$. Show that if

$$\mathbf{u} \cdot \nabla Q = u_r \frac{\partial Q}{\partial r} + u_z \frac{\partial Q}{\partial z} = 0$$

Then Q is a function of ψ alone, $Q = F(\psi)$.

(c) Applied to Bernoulli's expression

$$\mathbf{u} \cdot \nabla(\frac{p}{\rho} + \frac{1}{2}q^2) = 0$$

show that

$$\frac{p}{\rho} + \frac{1}{2}(u_r^2 + u_\theta^2 + u_z^2) = H(\psi)$$

for some function H. Applied to the equation for u_θ, show that $r u_\theta = C(\psi)$ for some function C. Show that this is a special case of Kelvin's theorem.

2. Continuing problem 1, we showed in class that

$$(u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z})(\omega_\theta / r) = \frac{2}{r^2} u_\theta \frac{\partial u_\theta}{\partial z}.$$

From this show (using results of problem 1) that ψ satisfies

$$L(\psi) = -CC_\psi + r^2 f(\psi)$$

for some functions C, f. Finally, show from the momentum equation that $f = H_\psi$ where H is defined in problem 1.

3. (Reading: Milne-Thomson p. 89, Batchelor p. 384) (a) Prove Kelvin's minimum energy theorem: In a simply-connected domain V let $\mathbf{u} = \nabla\phi, \nabla^2\phi = 0$, with $\partial\phi/\partial n = f$ on the boundary S of V. (This $\mathbf{u}$ is unique in a simply-connected domain). If $\mathbf{v}$ is any differentiable vector field satisfying $\nabla \cdot \mathbf{v} = 0$ in V and $\mathbf{v} \cdot \mathbf{n} = f$ on S, then

$$\int_V |\mathbf{v}|^2 dV \geq \int_V |\mathbf{u}|^2 dV.$$

(Hint: Let $\mathbf{v} = \mathbf{u} + \mathbf{w}$, and apply the divergence theorem to the cross term.)