Applied Mathematics II **PROBLEM SET 4** **Due February 27, 2003**

1. Solve, using characteristics, the following problem for a nonlinear wave equation:

$$u_t + u^2 u_x = 0,\ u(x_0, 0) = \begin{cases} 0, & \text{if } x_0 < 0, \\ x_0, & \text{if } x_0 \geq 0. \end{cases}$$

Show that u is constant on a characteristic and that the characteristics are straight lines. Then find $u(x,t)$ as an explicit function of x, t for $t > 0$. Check your answer by differentiation.

2. Consider

$$u_t + u^3 u_x = 0,\ u(x_0, 0) = \begin{cases} 0, & \text{if } x_0 < 0, \\ 1, & \text{if } x_0 \geq 0. \end{cases}$$

Indicate the structure of the solution in the x, t plane, indicating in particular the function $u = F(x/t)$ which holds in the expansion fan.

3. Devise a model for a one-way road which changes from one to two lanes at $x = 0$. Both density and velocity will generally change at such a lane change. Assume $\rho = \rho_1 < \rho_m$ and $u = u_1 = u_m(1 - \rho_1/\rho_m)$ for $x < 0$. Assume that as drivers cross $x = 0$, one-half of them instantaneously switch lanes and take up the appropriate new velocity. This means that the drivers will then move with a speed $u_2 = u_m(1 - \frac{1}{2}\rho_2/\rho_m)$, where ρ_2 is the density of the two-lane road, i.e. the number of cars per unit length including both lanes. Explain this formula for u_2. Determine the new constant density ρ_2 in $x > 0$ from the conservation of traffic flux at $x = 0$, namely $\rho_1 u_1 = \rho_2 u_2$. Reject any solution which gives $\rho_2 > \rho_m$ as unphysical. Show that $u_2 > u_1$. Make a plot of u_2/u_1 as a function of ρ_1/ρ_m for $0 < \rho_1 < .9$, to show the effect of lane addition on traffic speed. How would you generalize this model to one lane into N lanes, $N > 2$?

4. Consider $u_t + uu_x = 0,\ u(x, 0) = e^{-x^2}$. We know that shock formation occurs at the earliest time and position where the Jacobian $\frac{\partial x}{\partial x_0}$, associated with the characteristic lines $x = x(x_0, t)$, vanishes. Find the time and position of shock formation in this IVP.

5. After a football game the exit traffic onto a single-lane one-way road build up then falls to zero. At time t=0 the density of the traffic is:

$$\rho(x, 0) = \begin{cases} 0, & \text{if } |x| > 2 \text{ miles}, \\ 100(x + 2) \text{ vehicles/mile}, & \text{if } 2 \leq x \leq 0, \\ 100(2 - x) \text{ vehicles/mile}, & \text{if } 0 \leq x \leq 2. \end{cases}$$

Assume our standard model with $u_m = 60$ mph and $\rho_m = 250$ vehicles/mile. (x is measured in miles and t in hours.)

(a) Using characteristics, map out the function $\rho(x,t)$, indicating the regions of simple waves and the position x_s, t_s and time of shock formation. Show that the shock forms at time $t = t_s = 1/48$ hours ,i.e. 75 seconds.

(b) Using the shock speed formula, compute the path of the shock, $x = \xi(t), t > t_s$. (You will have to solve a first-order ODE, of a kind discussed in class.)

(c) Make a careful sketch of characteristics pattern in the $(x, 60t)$ plane, indicating the position of the point of initial shock formation and indicating roughly the path of the shock.