1. (a) If f, g are two twice differentiable functions in a domain D, prove Green's identity

$$\int_D f\nabla^2 g = g\nabla^2 f \ \ dV = \int_{\partial D} f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \ \ dS$$

(b) Let D be the a sphere of radius R_0 at the origin, f a harmonic function in D, and $g = \frac{1}{4\pi}(\frac{1}{R_0} - \frac{1}{R})$ where $R^2 = x^2 + y^2 + z^2$. Using the fact that $\nabla^2 \frac{1}{R} = -4\pi\delta(\mathbf{x}), \delta =$ delta function, show that the average of a harmonic function over a sphere is equal to the value of the function at the center of the sphere (here $f(0)$).

2. In the Butler sphere theorem, we needed the following result: Show that $\Psi_1(R,\theta) \equiv \frac{R}{a}\Psi(\frac{a^2}{R},\theta)$ is the streamfunction of an irrotational, axisymmetric flow in spherical polar cordinates, provided that $\Psi(R,\theta)$ is such a flow. (Hint; Show that $L_R\Psi_1(R,\theta) = L_{R_1}\Psi(R_1,\theta)$, where $R_1 = a^2/R$. Here $L_R\Psi = R^2\frac{\partial^2\Psi}{\partial R^2} + \sin\theta\frac{\partial}{\partial\theta}\big(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\big)$.)

3. Show that in spherical polar coordinates, the streamfunction Ψ for a source of strength Q, placed at the origin, normalized so that $\Psi = 0$ on $\theta = 0$, is given by $\Psi = \frac{Q}{4\pi}(1 - \cos\theta)$. (Recall $u_R = \frac{1}{R^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, u_\theta = \frac{-1}{R\sin\theta}\frac{\partial\Psi}{\partial R}$.) Find the streamfunction in spherical polars for the airship model consisting of equal source and sink of strength Q, the source at the origin and the sink at $R = 1, \theta = 0$, in a uniform stream with streamfunction $\frac{1}{2}UR^2\sin\theta^2$. The sink will thus involve the angle with respect to $R = 1, \theta = 0$. Use the law of cosines ($c^2 = a^2 + b^2 - 2ab\cos\theta$ for a triangle with θ opposite side c) to express Ψ in terms of R, θ.

4. (a) Show that the complex potential $w = Ue^{i\alpha}z$ determines a uniform flow making an angle α with respect to the x-axis and having speed U.

(b) Describe the flow field whose complex potential is given by

$$w = Uze^{i\alpha} + \frac{Ua^2e^{-i\alpha}}{z}.$$

5. Recall the following rule for the motion of point vortices in two dimensions: Each vortex moves with the velocity equal to the sum over the velocities contributed by all other vortices, at the point in question. That is, now using the complex potential.

$$dz_k(t)/dt = \overline{w'(z_k)}, w_k = \sum_{j=1, j\neq k}^{N} \gamma_j \log{(z - z_j(t))}, \gamma = -i\Gamma/2\pi,$$

where the strengths are γ_i and the positions are $z_i(t)$. (a) Using this rule, show that two vortices of equal strengths rotate on a circle with center at the midpoint of the line joining them, and find the speed of their motion in terms of γ and the separation distance.

(b) Show that two vortices of strengths γ and $-\gamma$ move together on straight parallel lines perpendicular to the line joining them. Again find the speed of their motion in terms of γ and separation distance.