Applied Mathematics II **PROBLEM SET 5** **Due March 6, 2003**

1. In class we considered $\rho_t + (1-2\rho)\rho_x = 0$ with $\rho_0 = \frac{2}{3}(1+x_0), -1 < x_0 < 0$ and otherwise zero. From the solution we found,

(a) sketch the density of the pack of cars as a function of x when $t = 1.5$.

(b) What is the equation satisfied by $x_c(t)$, the position of an individual car, then the car is within the triangle formed by the characteristics emanating from $x_0 = 0$ and $x_0 = 1$?

(c) Suppose, instead of $u(\rho) = (1-\rho)$ as in (a,b), we had taken $u = 1 - 2\rho/3$. What would the ODE problem for the shock path be in that case?

2. For the "red-then-green" stop-light problem discussed in class, let $\rho_0 = 1/4$ and $t_{G_1} = 2$.

(a) Sketch $\rho(x,1)$ and $\rho(x,3)$ based on the solution outlined in the handout.

(b) Verify from the shock path the statement that the shock will cross $x = 0$ at $t = t_{R_2}$ provided that $\rho_0 = 1/4$.

3. Determine the characteristic solution and fit the shock into the solution:

$$u_t + uu_x = 0, u(x_0, 0) = x_0 + 5, -5 < x_0 < 0, = \frac{5}{2}(2-a), 0 < a < 2, = 0, x < -5, x > 2.$$

Sketch your solution in the x, t plane. Give explicit expressions for the position of the shock, and the jump in u across the shock, as functions of time.

4. A linear vibrating string is modified by attaching it to a bed of springs which exerts a force proportional to the displacement of a point of the string from a stretched-straight position $u(x,t) = 0$. The equation for the displacement $u(x,t)$ is then

$$u_{tt} - c^2 u_{xx} + \lambda^2 u = 0,$$

where $c^2 = T/\rho, \lambda$ are constants. Solve the initial value problem for this equation given that

$$u(x,0) = \int_0^\infty e^{-k^2} e^{ikx} dk, \quad u_t(x,0) = 0.$$

Study the structure of the solution for large x, t using the method of stationary phase. Verify that the group velocity lies between $-c$ and $+c$. Focus on the waves moving to the right $(x, t > 0)$ and give an explicit expression for $\sqrt{t}u(x,t)$ as a function of x/t for $x, t \to \infty, 0 < x/t < 1$.