Prof. Childress

1 (a) $\frac{dx}{dt} = 2x \quad \frac{dy}{dt} = x - 3y$

The matrix

$$A = \begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}$$

Solving for eigenvalues,

$$\begin{pmatrix} 2-\lambda & 0 \\ 1 & -3-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(-3-\lambda)$$
$$= \lambda^2 + \lambda - 6$$
$$= (\lambda+3)(\lambda-2)$$

1 ✓

EIGENVALUES $= -3, 2$

Since one of the eigenvalues is > 0, the equilibrium $(0,0)$ is unstable. ✓

(b) SOLVING FOR EIGENVECTORS:

$$\vec{\lambda}_+ = \begin{pmatrix} 0 & 0 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \lambda_- = \begin{pmatrix} 5 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$x - 5y = 0 \qquad 5x + 0y = 0$$

$$x = 5y \qquad x + 0y = 0$$

1 $\vec{\lambda}_+ = \begin{pmatrix} 1 \\ 1/5 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ ✓ $\quad \lambda_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ✓ SINCE $y \in \mathbb{R}$

GENERAL SOLUTION:

$$c_1 \begin{pmatrix} 5 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$$

(c) PHASE PLANE EQ IS:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{x - 3y}{2x}$$

TAKING $y(x) = \frac{1}{5}x + s(x)$,

$$y'(x) = \frac{1}{5} + s'(x)$$

PLUGGING $y'(x)$ & $y(x)$ into above eq. ✓

$$\frac{1}{5} + s'(x) = \frac{x - 3\left[\frac{1}{5}x + s(x)\right]}{2x}$$ ✓

$$2x\left(\frac{1}{5} + s'(x)\right) = x - \frac{3}{5}x - 3s(x)$$

$$\frac{2}{5} + 2s'(x) = 1 - \frac{3}{5} - \frac{3s(x)}{x}$$ ✓

1 $$\frac{s'(x)}{s(x)} = \frac{-3}{2x}$$ ✓

$$\ln(s(x)) = -\frac{3}{2}\ln x + c$$

$$s(x) = x^{-3/2}e^{c}$$ ✓

$$\therefore \quad y(x) = \frac{1}{5}x + cx^{-3/2}$$ ✓

(d)

FROM PART B:

$$x = 5c_1e^{2t} \qquad y = c_1e^{2t} + c_2e^{-3t}$$

$$\frac{x}{5c_1} = e^{2t}$$

SUBSTITUTING FOR e^{2t} in the expression for y yields:

1 ✓

$$y = c_1 e^{2t} + c_2\left(e^{2t}\right)^{-3/2}$$

$$y = c_1\left(\frac{x}{5c_1}\right) + c_2\left(\frac{x}{5c_1}\right)^{-3/2}$$

$$y = \frac{1}{5}x + \underbrace{\frac{c_2}{(5c_1)^{-3/2}}}\, x^{-3/2}$$

MAKE $C = \frac{c_2}{(5c_1)^{-3/2}}$

$$y = \frac{1}{5}x + Cx^{-3/2}$$

(2) $\frac{dx}{dt} = x[a - b(x+y)]$ GIVEN: $\frac{c}{d} > \frac{a}{b}$ OR $\frac{d}{c} < \frac{b}{a}$

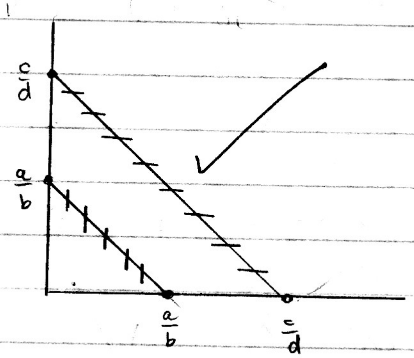
$\frac{dy}{dt} = y[c - d(x+y)]$

SOLVE FOR ISOCLINES FIRST:

$a - b(x+y) = 0$ $\quad\quad$ $c - d(x+y) = 0$

$a - bx - by = 0$ $\quad\quad$ $c - dx - dy = 0$

$y = \frac{a}{b} - x$ $\quad\quad$ $y = \frac{c}{d} - x$



SOLVING FOR X INTERCEPTS YIELD

$x = \frac{a}{b} - y$ $\quad\quad$ $x = \frac{c}{d} - y$

WHERE $\frac{dx}{dt} = 0$ $\quad\quad$ where $\frac{dy}{dt} = 0$

1

SINCE $\frac{c}{d} > \frac{a}{b}$, it is easy to plot the isoclines

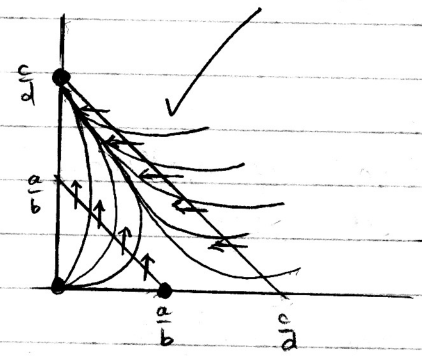
NEXT SOLVE FOR DIRECTION,

$\frac{dx}{dt} = a[1 - \frac{b}{a}(x+y)]$

$\frac{dy}{dt} = c[1 - \frac{d}{c}(x+y)]$

SINCE $\frac{d}{c} < \frac{b}{a}$, $\frac{dy}{dt}$ will increase & $\frac{dx}{dt}$ will decrease

1



TO FIND EQ. POINTS, WE HAVE TO SOLVE FOR ALL VALUES S.T. $\frac{dx}{dt} = \frac{dy}{dt} = \emptyset$.

SUCH VALUES ARE $(0,0)$ $(0, \frac{c}{d})$ & $(\frac{a}{b}, 0)$, indicated by dots.

$(0,0)$ is UNSTABLE

$(0, \frac{c}{d})$ is STABLE

$(\frac{a}{b}, 0)$ is UNSTABLE

OVER TIME, x is extinct & y prospers.

(a) $\frac{dN_1}{dt} = N_1(a - bN_1 - kN_2)$
$\frac{dN_2}{dt} = N_2(c - dN_2 - \sigma N_1)$

$6N_1 + N_2 = 2$

$2N_1 + N_2 = 1$

(b) SKETCH THE PHASE PLANE

ISOCLINES WHERE $\frac{dN_1}{dt} = 0$,

$6N_1 + N_2 = 2$

$N_2 = 2 - 6N_1$ N_2-intercept at 2

$6N_1 + N_2 = 2$

$N_1 = \frac{1}{3} - \frac{1}{6}N_2$ N_1-intercept at $\frac{1}{3}$ ✓

1

THE ISOCLINE N_2 & N_1 intercepts can be solved similarly, thus yielding respectively

1 & ½

(b) SHOWING NON-ZERO EQ POINT

SOLVING ABOVE SYSTEM YIELDS POINT:

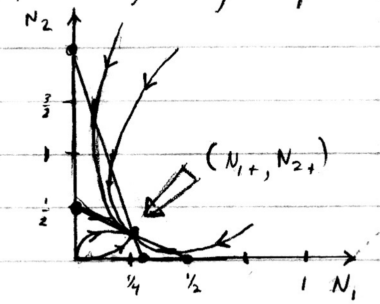
$$6N_1 + N_2 = 2$$
$$\underline{-6N_1 - 3N_2 = -3}$$
$$-2N_2 = -1$$
$$N_2 = \frac{1}{2}$$

1

PLUGGING N_2 into first eq.

GIVES: $N_1 = \frac{1}{4}$

$(N_1*, N_2*) = (\frac{1}{4}, \frac{1}{2})$ ✓



$N_2 \times N_1$ ax

(c) Find eigenvalues

$N_1 = \delta N_1 + \frac{1}{4}$ $N_2 = \delta N_2 + \frac{1}{2}$

$$\frac{d\delta N_1}{dt} = (\delta N_1 + \tfrac{1}{4})\left[2 - 6(\delta N_1 + \tfrac{1}{4}) - (\delta N_2 + \tfrac{1}{2})\right]$$
$$= (\delta N_1 + \tfrac{1}{4})\left[2 - \tfrac{4}{2} - 6\delta N_1 - \delta N_2\right]$$
$$= -\tfrac{6}{4}\delta N_1 - \tfrac{1}{4}\delta N_2 - \underbrace{\delta^2 N_1 N_2 - \delta^2 N_1^2}_{\text{NEGLECT}}$$

$$\frac{d\delta N_1}{dt} = -\tfrac{6}{4}\delta N_1 - \tfrac{1}{4}\delta N_2$$

1

$$\frac{d\delta N_2}{dt} = (\delta N_2 + \tfrac{1}{2})\left[1 - 2(\delta N_1 + \tfrac{1}{4}) - (\delta N_2 + \tfrac{1}{2})\right]$$
$$= (\delta N_2 + \tfrac{1}{2})\left[1 - \tfrac{1}{2} - \tfrac{1}{2} - 2\delta N_1 - \delta N_2\right]$$
$$= -\delta N_1 - \tfrac{1}{2}\delta N_2 - \underbrace{2\delta^2 N_1 N_2 - \delta^2 N_2^2}_{\text{NEGLECT}}$$

$$\frac{d\delta N_2}{dt} = -\delta N_1 - \tfrac{1}{2}\delta N_2$$ ✓

$$A = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{4} \\ -1 & -\frac{1}{2} \end{pmatrix}$$ ✓

$$\det(A - \lambda I) = 0$$
$$(-\tfrac{3}{2} - \lambda)(-\tfrac{1}{2} - \lambda) - \tfrac{1}{4} = 0$$
$$\lambda^2 + 2\lambda + \tfrac{3}{4} - \tfrac{1}{4} = 0$$
$$\lambda^2 + 2\lambda + \tfrac{1}{2} = 0$$

$a = 1$
$b = 2$
$c = \frac{1}{2}$

$$\frac{-2 \pm \sqrt{4 - 4(1)(\frac{1}{2})}}{2(1)}$$

$$\lambda_{1,2} = -1 \pm \frac{\sqrt{2}}{2}$$
$$\lambda_1 = -1 + \frac{\sqrt{2}}{2} = -.292$$
$$\lambda_2 = -1 - \frac{\sqrt{2}}{2} = -1.707$$

SINCE $\lambda_{1,2} < 0$, ✓ the point (N_{1*}, N_{2*}) is stable.