Applied Mathematics PROBLEM SET 6 Due March 13, 2003

1. Let $f(x)$ have Fourier transform $\hat{f}(k)$, that is $\hat{f}(k) = \mathcal{F}(f)$. Show that, if $\hat{f}(k) \to 0$ as $k \to \pm\infty$, that $\mathcal{F}^{-1}(\hat{f}'(k)) = -ix\mathcal{F}^{-1}(\hat{f})$. From this deduce that $xf(x)$ has the Fourier transform $-i\hat{f}'(k)$. Use this to find the Fourier transform of $xe^{-\alpha|x|}$, $\alpha > 0$.

2. Find the Fourier transforms with respect to x of the following differential equations:

$$(a)\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} - \frac{\partial^3 f}{\partial x^3} = 0,$$

$$\frac{\partial f}{\partial t} + x\frac{\partial f}{\partial x} = 0.$$

In (b) make use of problem (1) above.

3. From the derivation of the fundamental solution (with k = 1) $T = \frac{1}{\sqrt{t}}F(\eta), \eta = x/\sqrt{t}, F = \frac{1}{2\sqrt{\pi}}e^{-\eta^2/4}$ we arrived at the equation

$$F_\eta + \frac{1}{2}\eta F = C = constant.$$

Show that, if the constant is *not* taken equal to zero, there is another solution

$$S(x,t) = \frac{1}{\sqrt{\eta}}G(\eta), G(\eta) = e^{-\eta^4/4}\int_0^\eta e^{u^2/4}du.$$

Dividing $\int_0^\eta$ into $\int_0^M + \int_M^\eta$ with $0 < M < \eta$, and using integration by parts, show how the G behaves as $\eta \to \infty$, and in particular that it decays algebraically like $\frac{1}{\eta}$, not exponentially.

4.Use the Fourier transform to find the fundamental solution of the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0.$$

Express you answer as an inverse Fourier transform. Show that the answer has the form $u = t^{-1/4}f(x/t^{1/4})$. Derive a differential equation for f from this form and show that it reduces to $\eta f - 4f\eta\eta\eta = 0$, given that F and its derivatives vanish at infinity. Show that the Fourier integral solution satisfies this differential equation.

5. Find the solution of the following inhomogeneous IBVP for the heat equation:

$$T_t - kT_{xx} = 1, 0 < x < \pi,$$

$$T(0,t) = 1, T(\pi,t) = 3, T(x,0) = 0.$$

Your solutions will be in the form

$$T = f(x) + \sum_{n=1}^{\infty} c_n e^{-n^2kt}\sin nx,$$

where the c_n should be given explicitly.

6. Suppose that the IVP for the 1D heat equation on $-\infty < x < +\infty$ is solved with $u(x,0) = f(x), f(-x) = -f(x)$. Show that the solution may be written in the form

$$u(x,t) = \int_0^{+\infty} f(\xi)[U(x-\xi,t) - U(x+\xi,t)]d\xi,$$

where U is the fundamental solution. From this deduce that u is the solution to the IVP for the half interval $x > 0$, with initial values $f(x)$ on this interval, subject to the boundary condition $u(0,t) = 0$. This is an example of *the reflection method*.