1. For $t \geq 0$ let $T(x,t)$ be a bounded solution of the heat equation $T_t - DT_{xx} = 0$ satisfying the initial condition $T(x,0) = f(x)$ on the infinite interval $(-\infty, +\infty)$. Let

$$T_1(x,t) = T(x,t) + \frac{x}{t^{3/2}} e^{-\frac{x^2}{4Dt}}.$$

(a) Show that T_1 also solves the heat equation for $t > 0$. (For the new term either verify directly or relate to the derivative of a fundamental solution.) (b) Show that

$$\lim_{t \to 0} T_1 = f(x).$$

(Hint: Check the two cases $x = 0$ and $x \neq 0$ separately.) (c) How does this solution contradict the conditions of the uniqueness theorem given in class? (Hint: check the curve $x = \sqrt{t}$.)

2. Show that
$$E(x,t) = Erf(\eta), \eta = x/\sqrt{4Dt}, Erf(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2} du$$

is a solution of the heat equation, whose derivative with respect to x is twice the fundamental solution. Show that E satisfies the following initial conditions on the interval $(-\infty, +\infty)$:

$$E(x,0) = 1, x > 0, = -1, x < 0.$$

Sketch $E(x,t)$ versus x for various times (qualitatively).

3. Using the Poisson representation, show that the solution of the IVP for the heat equation on $(-\infty, +\infty)$ with initial condition $T(x,0) = 1, |x| < 1, = 0$ otherwise, is given by

$$T(x,t) = \frac{1}{2}[Erf(\frac{x+1}{\sqrt{4Dt}}) - Erf(\frac{x-1}{\sqrt{4Dt}})]$$

where Erf is the function defined in problem 3. Verify from the solution that the initial conditions are indeed satisfied.

4. Using the similarity method, derive the fundamental solution $U(x,y,t)$ of the heat equation in 2D, satisfying
$$U_t - D(U_{xx} + U_{yy}) = 0, \ t > 0,$$
$$\lim_{t \to 0+} U = \delta(x)\delta(y).$$

Note , you may assume $U = U(r,t), r^2 = x^2 + y^2$, and $U_{xx} + U_{yy} = U_{rr} + r^{-1}U_r$, so that the integral condition is

$$2\pi \int_0^\infty Urdr = 1, \ t > 0.$$

Also, we may assume U and its derivatives vanish at infinity. Show that

$$U = \frac{1}{4\pi kt} e^{-r^2/(4Dt)}.$$

Use the solution to write the solution to th IVP in the Cauchy form for 2D.

5. Solve the following equilibrium problem in three dimensions. Heat is being generated within the spherical annulus $a < r < b$ at a constant rate q_0. That is

$$-K(T_{rr} + \frac{2}{r}T_r) = q_0, a < r < b.$$

Here the material within the annulus has conductivity K. The inner spherical surface is the boundary of an *insulator*, so $T_r(a) = 0$. The outer spherical surface is held at temperature T_0. What is the temperature distribution, and what is the heat flux through the spherical surface $r = b$? Interpret this last result in in physical terms as conservation of heat?

6. The ocean may considered for this problem a 2D domain where contaminants such as oil from an oil spill diffuse with difusivity k. Suppose that a spill of volume Q occurs at $t = 0$ at a point $(0, y_0)$ where $y_0 > 0$ and the line $y = 0$ is a coastline. The spill is cleaned as it arrives at the coast, so that the flux of oil onto the beach is given by

$$F = \int_{\infty}^{+\infty} -D\frac{\partial u}{\partial y}(x, 0, t)dx,$$

where $u(x, y, t)$ is the oil density, solving the diffusion equation in 2D with diffusivity D. Give an expresssion for F as aa function of time. At what time, in terms of y_0, k, does the flux reach a maximum?. Use the fundamental solution from problem 4.