Contour integrals by Laurent series

This was touched on at the end of lecture 8. The idea is that if we know the Laurent expansion then from the expressions for the coefficients of the expansion as contour integrals we know how to do some contour integrals.

Example: Find

$$\oint \frac{1}{z^2 \sin z} dz$$

taken in the positive direction. Now by formulas for the coefficients in the Laurent expansion we know this equals $2\pi i$ times the coefficient a_1 in the Laurent expansion of $1/\sin z$ valid for $|z| > 0$. Now

$$\frac{1}{\sin z} = \frac{1}{z(1 - \frac{1}{6}z^2 + \ldots)} = \frac{1}{z}(1 + \frac{1}{6}z^2 + \ldots).$$

So $a_1 = 1/6$, and

$$\oint \frac{1}{z^2 \sin z} dz = 2\pi i/6 = \pi i/3.$$

We can write this as

$$\oint \left[\frac{1}{z^3} + \frac{1}{6z} + \ldots\right] dz = 2\pi i/6 = \pi i/3.$$

The integral picks out the coefficient of the $\frac{1}{z}$ term. This is an example of the method of residues, although here we have simply made use of the Laurent expansion theorem.