Some remarks on finding expansions of periodic solutions and fundamental solution matrices for Mathieu's equation

Mathieu's equation can be uses as an example of finding solutions with are periodic with the same period as the coefficient. Consider the equation in the form

$$z_{tt} + (a + b\cos 2t)z = 0.$$

We can try to find values of a, b for which periodic solutions with period π exist. To illustrate this in a perturbative scheme consider the special case $a = 4n^2 + \gamma(\epsilon), b = \epsilon$, n an integer. Solutions can be represented as power series in ϵ, of the form

$$z(t) = z_0(t) + \epsilon z_1(t) + \epsilon^2 z_2(t) + \ldots.$$

In general, to find a periodic solution we must allow γ to have a power series at our disposal, with $\gamma_0 = 0$:

$$\gamma(\epsilon) = \gamma_1\epsilon + \gamma_2\epsilon^2 + \ldots.$$

The equations for the term in sequence are

$$z_0'' + 4n^2 z_0 = 0$$
$$z_1'' + 4n^2 z_1 = -\gamma_1 z_0 - \cos 2t\ z_0$$
$$z_1'' + 4n^2 z_1 = -\gamma_1 z_1 - \gamma_2 z_0 - \cos 2t\ z_1$$

and so on. To illustrate the procedure consider the case $n = 1$ Suppose we look for a solution satisfying $z(0) = 1, z'(0) = 0$. Then the intitial conditions determine $z_0 = cos 2t$, and

$$z_1'' + 4z_1 = -\gamma_1\cos 2t - \cos^2 2t\ z_0$$

The last term generates the particular solution $-\frac{1}{8} + \frac{1}{24}\cos 4t$, and the term with γ_1 generates the particular solution $\frac{-\gamma_1}{4} t\sin 2t$. This is a *resonance term* which will not allow the solution to be periodic. Hence we must set $\gamma_1 = 0$. To satisfy the appropriate conditions $z_k(0) = z_k'(0) = 0, k \geq 1$ we then must add a solution of the homogeneous equation, to obtain

$$z_1 = -\frac{1}{8} + \frac{1}{24}\cos 4t + \frac{1}{12}\cos 2t.$$

In a similar way we solve for z_2 but now γ_2 must be chosen non-zero to remove the possibility of resonance terms. We find

$$\gamma_2 = \frac{5}{48}, z_2 = \frac{-1}{96} + \frac{1}{12\cdot 244}\cos 4t + \frac{1}{32\cdot 48}\cos 6t + \frac{29}{48\cdot 96}\cos 2t.$$

Note that this solution is even in t. We can similarly find periodic solutions odd in t, satisfying $z(0) = 0, z'(0) = 1$, with the expansions

$$\gamma = -\frac{1}{48}\epsilon^2 + \ldots, z = \frac{1}{2}\sin 2t + \epsilon[\frac{1}{48}\sin 4t - \frac{1}{24}\sin 2t] + \frac{\epsilon^2}{96}[\frac{1}{32}\sin 6t - \frac{1}{6}\sin 4t + \frac{23}{96}\sin 2t] + \ldots.$$

Since γ has a different variation with ϵ for these two solutions, only one periodic solution obtains for any given value of ϵ. For any given equation of this form a second linearly independent solution can be found by variation of parameters, or perturbatively by the process just described, but now with the prescribed $\gamma(\epsilon)$. As we have seen, without control over γ we will introduce resonance terms, and the solution will not be periodic. You can check that this is just an example of case (iii) for Hill's equation, wherein the the descriminant is 0 and we cannot decide boundedness (here periodicity) without additional knowledge about the equation.

That is, if were look for a fundamental solution matrix Z such that $Z(0) = I$, *and* we stipulate that $\gamma = \frac{5}{48}\epsilon^2 + \ldots$, then $Z = [z^(1) \quad z^{(2)}]$ where $z^{(1)}$ is the first solution computed above. However $z^{(2)}$ differs from the second solution above in that there is now a resonance term. We find

$$z^2 = \frac{1}{96}[\frac{1}{32}\sin 6t - \frac{1}{6}\sin 4t + \frac{11}{4}t\cos 2t - \frac{91}{96}\sin 2t].$$

The resonance term is then responsible for the non-diagonal term $z^{(2)}(\pi)$ in Ω:

$$Z(\pi) = \Omega = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad \lambda = \frac{11\pi\epsilon^2}{384}$$

This is an instance of case (iii) for Hill's equation; Ω is easily brought into Jordan Normal Form .