Ordinary Differential Equations   Homework 1   Childress, Spring 2002

Due January 29

1. The *logistic* equation $\frac{dy}{dx} = ry(k - y)$ is often used as a a model for population growth. Here r, k are positive contants, and we take $y \geq 0$.

(a) Sketch the direction field for the equation, indicating the form of integral curves intersecting the y-axis both above and below $y = k$.

(b) Integrate to find the general solution of the initial value problem $y(0) = y_0 > 0$.

2. The *Ricatti* equation has the general form $\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x)$. This equation cannot in general be integrated. However it can be reduced to a linear second-order equation. (a) Show that this is possible by the substitution $y = -a^{-1}u^{-1}du/dx$ and find the form of the equation. Find the solution of the resulting equation when $a = e^x, b = x, c = 0$ and $y(0) = y_0$.

3. The equation of motion of a point $\mathbf{r}(t) = (x(t), y(t), z(t))$ in rotation with fixed angular velocity $\omega = (\omega_1, \omega_2, \omega_3)$ satisfied the third-order linear system

$$\frac{d\mathbf{r}}{dt} = \omega \times \mathbf{r}. \tag{1}$$

(a) Show that for any solution $\mathbf{r}(t)$ of (1) necessarily $x^2 + y^2 + z^2 = R^2 = \text{constant}$.

(b) Show that the transformation $\lambda = \frac{x+iy}{R-z}$ (Stereographic projection) from the sphere $x^2 + y^2 + z^2 = R^2$ to the complex λ plane maps (1) into a complex valued Ricatti equation for $\lambda(t)$.

4. Discuss the existence and uniqueness of solutions of the initial value problem

$$\frac{dy}{dx} = y^{1/2} + y^{3/2}, \;\; y(0) = y_0 \geq 0.$$