Ordinary Differential Equations   Homework 10   Childress, Spring 2002

Due April 16

Note: This is the last homework to be handed in, graded, and returned with answers. Problems to be handed April 16 and 23 will constitute a take-home final, due in my office on or before May 2.

1. Consider complex-valued scalar functions of the real variable x, the second-order differential operator $Ly = (p(x)y')' - q(x)y$, and the boundary conditions $\alpha y(a) + \beta y'(a) = 0, \gamma y(b) + \delta y'(b) = 0$. The inner product here is

$$(u, v) = \int_a^b \overline{v} u dx,$$

where the bar denotes complex conjugate. Also p is continuously differentiable, q continuous, and $p \neq 0$ on $[a, b]$. Show that this problem is self-adjoint if and only if p and q are real, and $\alpha\overline{\beta} = \overline{\alpha}\beta$ and $\gamma\overline{\delta} = \overline{\gamma}\delta$, (which means $\alpha, \beta, \gamma, \delta$ can be taken as real).

2. Considering, as in problem 1 above, complex-valued functions of x on $[a, b]$, under what conditions on $p(x), q(x)$ will the differential operator $Ly = p(x)y'' + q(x)y'$ be formally self-adjoint?

3. Prove that the following BVP has only the trivial solution among function twice continuously differentiable functions $y(x)$ defined on $[0, 1]$:

$$(x^2 y')' - e^{-x} y = 0, y'(1) + y(1) = 0, y(0) = 0.$$

4. For the operator $L = \frac{d^2}{dx^2}$ and $y(x)$ defined on $[0, 1]$, with the boundary conditions $y(0) = 0$, $y'(1) = 0$, we consider the construction of the Green's function $K(x, \xi)$ and the eigenvalues and eigenfunctions satisfying $Ly - \lambda y = 0$.

(a) Show that 0 is not an eigenvalue.

(b) Construct explicitly $K(x, \xi)$.

(c) Find all non-zero eigenvalues, and the corresponding eigenfunctions.

(d) Represent K in terms of the eigenfunctions and eigenvalues.

(e) Recover the identity $\pi^2/8 = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

5. Consider the equation

$$Ly = x(xy')' + 3y$$

and the eigenvalue problem $Ly = \lambda y, 1 \leq x \leq 2$, with boundary conditions $y(1) = y(2) = 0$.

(a) Show that L is self-adjoint on functions satisfying these conditions provided that we define the inner product by $(z, y) = \int_1^2 \frac{1}{x} zy dx$.

(b) Find the orthonormal (in the above inner product) eigenfunction and eigenvalues satisfying $Ly_n = \lambda_n y_n,\ y_n(1) = y_n(2) = 0, n = 1, 2 \ldots$, and define $K = \sum_{n=1}^{\infty} y_n(x) y_n(\xi)/\lambda_n$. (Hint: $x^i = e^{i \ln x}$.)

(c) Express the solution to the inhomogeneous equation $Lf = \log x$ as an expansion in the eigenfunctions.

(d) Solve $Lf = \log x$ explicitly. (Hint: $\log x/3$ is a particular solution.)